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Vectorization of set-valued maps with respect to total ordering cones and its applications to set-valued optimization problems

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ABSTRACT

As a result of our previous studies on finding the minimal element of a set in n-dimensional Euclidean space with respect to a total ordering cone, we introduced a method which we call "The Successive Weighted Sum Method" (Küçük et al., 2011 [1,2]). In this study, we compare the Weighted Sum Method to the Successive Weighted Sum Method. A vector-valued function is derived from the special type of set-valued function by using a total ordering cone, which is a process we called vectorization, and some properties of the given vector-valued function are presented. We also prove that this vector-valued function can be used instead of the set-valued map as an objective function of a set-valued optimization problem. Moreover, by giving two examples we show that there is no relationship between the continuity of set-valued map and the continuity of the vector-valued function derived from this set-valued map.

1. Introduction

The main purpose of vector optimization problems is to find optimal elements of a given set in partially ordered linear spaces. A set-valued optimization problem is an extension of a vector optimization problem.

The set-valued optimization methods and their applications in parallel with the methods and applications of vector optimization have been in the spotlight in last few decades. Recent developments on vector and set-valued optimization can be found in [3–9].

Scalarization methods convert vector or set-valued problems into real-valued problems. Scalarization is used for finding optimal solutions of vector-valued optimization problems in partially ordered spaces.

A construction method of an orthogonal base of $\mathbb{R}^n$ and total ordering cones on $\mathbb{R}^n$ using any nonzero vector in $\mathbb{R}^n$ was given in [1]. A solution method for vector- and set-valued optimization problems with respect to a total ordering cone by using scalarization was also given in the same study.

In this paper, we first give basic definitions and theorems, followed by an example and comparison of the Weighted Sum Method and the Successive Weighted Sum Method. We continue with a proof of the theorem which gives the relationship between the solutions of the methods, and show the existence of vector-valued function derived from a given set-valued map. Then, the order relation between set-valued maps is provided by utilizing the order of the vector-valued functions. By using these vector-valued functions, we also show that the minimal element of the vector optimization problem with respect to the total ordering cone is the minimal element of the given set-valued optimization problem. Finally, we show that there is no relationship between the continuity of a set-valued map and the continuity of the vector-valued map derived by utilizing this set-valued map.

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2. Mathematical preliminaries

In this section, we provide some basic notations, definitions and theorems.

Given a vector space Y, any relation which is reflexive, anti-symmetric, transitive, compatible with addition and compatible with scalar multiplication is called a partial order on Y. For a pointed (i.e., $C \cap (-C) = \{0\}$) convex cone $C \subset Y$ the relation defined by

$$y_1 \leqslant_C y_2 \quad \Leftrightarrow \quad y_2 \in y_1 + C, \quad \text{for all } y_1, y_2 \in Y$$

is a partial order on Y.

Moreover, if a partial order compares any two vectors in Y, it is called a total order. In this paper we mainly work on total orders. So, we need some important properties of them (for proofs of the following properties and additional information one can see [1]). If a pointed, convex ordering cone K satisfies

$$K \cup (-K) = Y$$

then "$\leqslant_K$" is a total order on Y.

For an ordered orthogonal set $\{r_1, r_2, \ldots, r_n\} \subset \mathbb{R}^n$, the set

$$K = \left[\bigcup_{i=1}^{n} \{r \in \mathbb{R}^n\text{: for all } j < i,\ \langle r_j, r\rangle = 0,\ \langle r_i, r\rangle > 0\}\right] \cup \{0\} \tag{1}$$

is a total ordering cone on $\mathbb{R}^n$ and every total order on $\mathbb{R}^n$ can be represented by such a cone.

Let C be a cone and $B \subset C$ be a convex set which does not contain 0. If for all $c \in C$ there exists a unique $b \in B$ and $\lambda > 0$ such that $c = \lambda b$ then B is said to be a base of cone C. In addition, if B is compact then, it is said that C has a compact base. The following theorem shows that every ordering cone with a compact base is included in the interior of some total ordering cone.

Theorem 2.1. *(See [1].) Let C be a cone in $\mathbb{R}^n$. If C has a compact base then there is a total ordering cone K such that*

$$C \setminus \{0\} \subset int(K).$$

The following theorem gives a property of total orders: The set of minimal elements of a given set, with respect to a total ordering cone, cannot have two or more elements.

Definition 2.2. Let Y be a vector space partially ordered by an ordering cone C, $A \subset Y$ be a nonempty set and $\bar{x} \in A$.

(i) If $(\{\bar{x}\} - C) \cap A = \{\bar{x}\}$ then $\bar{x}$ is said to be a minimal element of A with respect to the ordering cone C. The set of all minimal elements of A with respect to C is denoted by $\min(A, C)$.

(ii) If $A \subset \{\bar{x}\} + C$ then $\bar{x}$ is said to be a strongly minimal element of A with respect to the ordering cone C.

Theorem 2.3. *(See [1].) Let K be a total ordering cone in $\mathbb{R}^n$. If a set $A \subset \mathbb{R}^n$ has a minimal element with respect to this cone then this minimal element is unique.*

Theorem 2.4 presents that minimality and strong minimality are the same with respect to a total ordering cone.

Theorem 2.4. *(See [1].) Let $K \subset \mathbb{R}^n$ be a total ordering cone, $A \subset \mathbb{R}^n$ and $\bar{x} \in A$. Then, $\bar{x}$ is a minimal element of A with respect to K if and only if $\bar{x}$ is a strongly minimal element of A with respect to K.*

3. The Successive Weighted Sum Method

In this section, we label the method given in [1] as the Successive Weighted Sum Method and study this method. We give an example to show differences between the Weighted Sum and the Successive Weighted Sum Method and we give a theorem that shows the relationship between their solutions.

Example 3.1. Let price/performance and the fuel consumption of the vehicles be as in Fig. 1. Since we prefer less price with respect to performance and less fuel consumption, we use $\mathbb{R}^2_+$ as the ordering cone and try to find the minimal elements. If we choose $(1, 1)$ as the vector r_1 in the Successive Weighted Sum Method, the solution set of the first step is the same as the solution set of the Weighted Sum Method when the weight vector is chosen as $w = r_1$. That is because we use the same scalarization for the first step. If we choose $r_2 = (1, -1)$ for the second step of the Successive Weighted Sum Method then we get the unique solution of the problem. So, A is the unique solution of this problem.

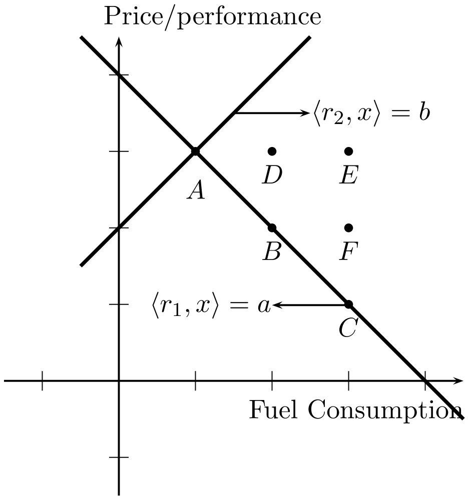

Fig. 1. Level sets for the Weighted Sum and the Successive Weighted Sum Scalarization.

In the Weighted Sum Method, the priorities of the consumer's criteria give us the weight vector. For example, if the second criterion is more important than the first, we can choose the vector $(1, 2)$ as the weight vector in order to emphasize the importance of the second criterion. The same prioritization can also be done in the Successive Weighted Sum Method. Moreover, priorities can be assigned to multiple criteria in such a way that the first vector r_1 is the most important criterion (i.e. has the highest priority), then comes the second vector r_2, then comes the third important criterion r_3, and so on. In the given example, we chose r_2 as $(1, -1)$, thus among the possible solutions of the first scalar problem, we preferred the vehicle that consumes less fuel.

In the Weighted Sum Method we obtain a set of solutions ensuring minimality of a scalar problem for a weight vector. In the Successive Weighted Sum method we get a solution vector ensuring minimality of n successive scalar problem for n linearly independent weight vectors. In the example, the solution set of the scalar problem

$$(SP_1) \quad \begin{cases} \min \langle r_1, f(x) \rangle \\ \text{s.t. } x \in X \end{cases}$$

is also a solution set for the Weighted Sum Method, if we choose weight vector $w = r_1$. This set is $\{A, B, C\}$. The solution of the second scalar problem

$$(SP_2) \quad \begin{cases} \min \langle r_2, f(x) \rangle \\ \text{s.t. } f(x) \in Sol(SP_1) \end{cases}$$

is the solution of the Successive Weighted Sum Method. This vector is A for $r_2 = (1, -1)$. If we choose $r_2 = (-1, 1)$ then the solution is the vehicle C.

As seen in [1] if the ordering cone is a total ordering cone, then the Successive Weighted Sum Method gives the solution of the problem. But, this cannot be obtained by using the Weighted Sum Method. That is because the solution to a problem with respect to a total ordering cone is unique for any given problem. In the previous example, if we choose the total ordering cone

$$K := \{y \in \mathbb{R}^2\colon \langle r_1, y \rangle > 0\} \cup \{y \in \mathbb{R}^2\colon \langle r_1, y \rangle = 0,\ \langle r_2, y \rangle > 0\} \cup \{0\}$$

then the solution of the problem is A. The Successive Weighted Sum Method ensures this solution by choosing the vectors $\{r_1, r_2\}$ respectively. But the Weighted Sum Method may not.

It is known that if the image set $f(X)$ is cone-convex then all minimal elements of the image set can be found by using the Weighted Sum Method [10]. That is not valid for the Successive Weighted Sum Method. In the example, the vehicle B cannot be the solution of the Successive Weighted Sum Method for any orthogonal set $\{r_1, r_2\}$. But it is a convex combination of the solutions of the Successive Weighted Sum Method for different orthogonal set of weight vectors (A and C). We give the following theorem guaranteeing this property.

Theorem 3.2. *Let X be a set and $f : X \to \mathbb{R}^n$ be a function. If the image of f is cone closed with respect to cone C with a compact base, i.e. $f(X) + C$ is closed where $C \subset \mathbb{R}^n$ is a cone with compact base, then the solution set of the vector optimization problem*

$$(VP) \quad \begin{cases} \min f(x) \\ s.t.\ x \in X \end{cases}$$

with respect to the Weighted Sum Method can be written as convex combinations of the solutions of (VP) which are obtained by applying the Successive Weighted Sum Method for ordered different orthogonal sets.

Proof. It is enough to show that any extreme point of the convex hull of the solution set of (*VP*) with respect to the Weighted Sum Method is also a solution of (*VP*) with respect to the Successive Weighted Sum Method for some ordered orthogonal set. First for a weight vector w the convex hull of the solution set for the Weighted Sum Method is in the form

$$A_1 := conv\big(\{y\colon\ y = f(x) \text{ for } x \in X,\ \langle w, y\rangle = a\}\big)$$

where a is a real number. Let $\bar{y}$ be an extreme point of the set A_1 and $r_1 = w$. Since $\bar{y}$ is an extreme point of A_1 it has a supporting hyperplane at $\bar{y}$. Let the vector for this hyperplane be r_2. It is obvious that r_2 is linearly independent of r_1 and we can choose r_2 as orthogonal to r_1. This hyperplane is in the form

$$H_1 := \big\{y \in \mathbb{R}^n\colon\ \langle r_2, y\rangle = \langle r_2, \bar{y}\rangle\big\}.$$

Then we get the set

$$A_2 := A_1 \cap H_1.$$

This set is also convex and since $\bar{y}$ is an extreme point of A_1 it is also an extreme point of A_2. Moreover, it has a supporting hyperplane at $\bar{y}$. Let the vector for this hyperplane be r_3 and we can choose this vector orthogonal to r_1 and r_2. Hence, the supporting hyperplane will be in the form

$$H_2 := \big\{y \in \mathbb{R}^n\colon\ \langle r_3, y\rangle = \langle r_3, \bar{y}\rangle\big\}.$$

The set

$$A_3 := A_2 \cap H_2$$

is also convex and $\bar{y}$ is an extreme point for this set. We can obtain an orthogonal set $\{r_1, r_2, \ldots, r_n\}$ by applying this procedure and $\bar{y}$ is the unique solution of n successive scalar problems

$$(SP_1) \quad \begin{cases} \min\big\langle f(x), r_1\big\rangle \\ \text{s.t. } x \in X \end{cases}$$

and for $i \in \{2, \ldots, n\}$

$$(SP_i) \quad \begin{cases} \min\big\langle f(x), r_i\big\rangle \\ \text{s.t. } f(x) \in A_{i-1}. \end{cases} \qquad \square$$

4. Vectorization

In this section, we present a method to obtain a vector-valued function from a cone-closed and cone-bounded set-valued function. A pre-order for sets was given by using ordering cones [11]. We show that using a total ordering cone, we can compare any two nonempty sets and this process gives us an order relation of sets. By using this order relation, we find the relationship between order relations of set-valued maps and vector-valued functions. Thus, it is obvious that we can get the same solution set for a given set-valued optimization problem and corresponding vector optimization problem. At the end of this section, two examples are given to show that the continuity of set-valued maps does not require the continuity of the corresponding vector-valued maps and vice versa.

In this study, we generally use cone-bounded and cone-closed sets. So we start with the definition of these concepts.

Definition 4.1. Let Y be a vector space ordered by the ordering cone C and let $A \subset Y$.

(i) If $A + C$ is closed then A is said to be C-closed.
(ii) If there exists a $y \in Y$ such that $A \subset y + C$ then A is said to be C-bounded.

Theorem 4.2. *Let X be a nonempty set and $C \subset \mathbb{R}^n$ be a cone with a compact base and nonempty interior $\operatorname{int} C$. If $F : X \rightrightarrows \mathbb{R}^n$ is a C-closed, C-bounded set-valued map then there exists a $V_F : X \to \mathbb{R}^n$ vector-valued function such that $\{V_F(x)\} = \min(F(x), K)$ for all $x \in X$ with respect to a total ordering cone K.*

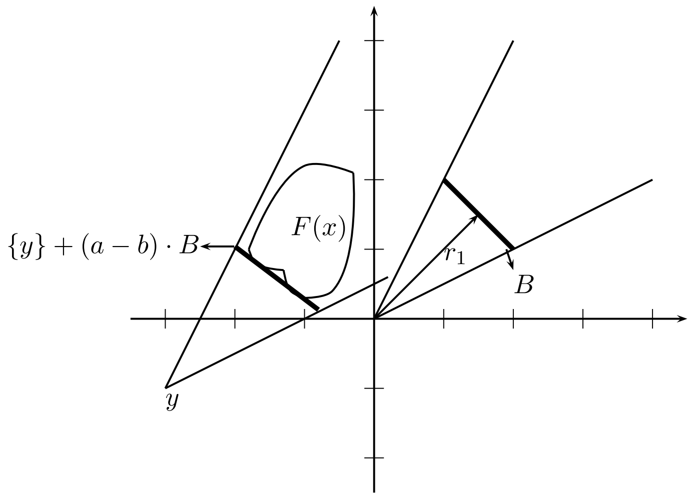

Fig. 2. The minimal elements of (SP_1).

Proof. Since the cone C has a compact base then there exist an orthogonal set $\{r_1,\dots,r_n\}$ and a total ordering cone K such that

$$K=\left(\bigcup_{i=1}^{n}\bigl\{a\in\mathbb{R}^n\colon \text{ for all } j<i,\ \langle r_j,a\rangle=0,\ \langle r_i,a\rangle>0\bigr\}\right)\cup\{0\},$$
$$C\subset\bigl\{r\in\mathbb{R}^n\colon \langle r_1,r\rangle>0\bigr\}\cup\{0\} \tag{2}$$

and $B:=\{c\in C\colon \langle r_1,r\rangle=1\}$ is a compact base of C.

For each $x\in X$ since $F(x)$ is C-bounded there exists an element $y\in\mathbb{R}^n$ such that

$$F(x)\subset\{y\}+C.$$

Namely, $y\leqslant_C\tilde{y}$, for all $\tilde{y}\in F(x)$. Since $r_1\in C^\sharp=\{x\in\mathbb{R}^n \mid \langle x,c\rangle>0 \text{ for all } c\in C\setminus\{0\}\}$, the function $\langle r_1,\cdot\rangle$ is strictly increasing with respect to the cone C [9]. Therefore, $\langle r_1,y\rangle\leqslant\langle r_1,\tilde{y}\rangle$, for all $\tilde{y}\in F(x)$. Hence, the set $\{\langle r_1,\tilde{y}\rangle\colon \tilde{y}\in F(x)\}$ is bounded from below.

Since the minimal elements of $F(x)$ are also the minimal elements of $F(x)+C$, $F(x)+C$ is closed and bounded from below. Let

$$a:=\min\bigl\{\langle r_1,\tilde{y}\rangle\colon \tilde{y}\in F(x)\bigr\}$$

and $b:=\langle r_1,y\rangle$. It is obvious that $b\leqslant a$.

The minimal elements of the scalar problem

$$(SP_1)\quad \begin{cases}\min\langle r_1,\tilde{y}\rangle\\ \text{s.t. } \tilde{y}\in F(x)\end{cases} \tag{3}$$

are in the form of

$$F(x)\cap\bigl((a-b)\cdot B+\{y\}\bigr)$$

(see Fig. 2).

Since the minimal elements of $F(x)$ and $F(x)+C$ are the same

$$\bigl(F(x)+C\bigr)\cap\bigl((a-b)\cdot B+\{y\}\bigr)=F(x)\cap\bigl((a-b)\cdot B+\{y\}\bigr).$$

Since $F(x)+C$ is closed and $((a-b)\cdot B+\{y\})$ is compact then the set

$$A_1:=F(x)\cap\bigl((a-b)\cdot B+\{y\}\bigr)$$

is also compact.

The minimal elements of the scalar problem

$$(SP_2)\quad \begin{cases}\min\langle r_2,\tilde{y}\rangle\\ \text{s.t. } \tilde{y}\in A_1\end{cases} \tag{4}$$

are in the form of

$$A_2 := A_1 \cap \{r \in \mathbb{R}^n\colon \langle r_2, r\rangle = c\}$$

where $c \in \mathbb{R}$. Since A_1 is compact and the hyperplane $\{r \in \mathbb{R}^n\colon \langle r_2, r\rangle = c\}$ is closed A_2 is nonempty and compact.

If we continue this process we obtain the scalar problem

$$(SP_i) \quad \begin{cases} \min\langle r_i, \tilde{y}\rangle \\ \text{s.t. } \tilde{y} \in A_{i-1} \end{cases} \tag{5}$$

for each $i \in \{3, \ldots, n\}$. The sets of minimal elements of these problems are nonempty and compact.

As K is a total ordering cone, the last set of minimal elements A_n has only one element. If we take

$$\{V_F(x)\} = A_n = \min(F, K)$$

then $V_F : X \to \mathbb{R}^n$ is a vector-valued function. □

Remark 4.3.

- The vector function $V_F(\cdot)$ derived in Theorem 4.2 is called K-minimal function of F.
- The K-minimal function of a set-valued map is unique depending on the total ordering cone K. For a given set-valued map and a chosen total ordering cone K we obtain a vector-valued function. If we choose another total ordering cone, we will get a different K-minimal function corresponding to the same set-valued map.

On the other hand, we can get the same K-minimal function for different set-valued maps by using the same total ordering cone K.

The following definition was given in [11].

Definition 4.4. Let $C \subset \mathbb{R}^n$ be an ordering cone and let A, B be any two C-bounded and C-closed nonempty subsets of $\mathbb{R}^n$. Then the relation $\leqslant_C$ is defined by

$$A \leqslant_C B \quad \Leftrightarrow \quad B \subset A + C.$$

Remark 4.5. The relation given in Definition 4.4 is reflexive, transitive, compatible with vector addition and scalar multiplication. But it is not anti-symmetric.

Proof. Let A, B, D be any C-bounded and C-closed nonempty subsets of $\mathbb{R}^n$.

(i) $A \subset A + C$ for any nonempty $A \subset \mathbb{R}^n$. Then $A \leqslant_C A$ and hence, this relation is reflexive.

(ii) Let $A \leqslant_C B$ and $B \leqslant_C D$ then $B \subset A + C$ and $D \subset B + C$. Since C is an ordering cone it is a convex cone then $C = C + C$. Therefore

$$D \subset B + C \subset (A + C) + C = A + (C + C) = A + C.$$

Hence the relation is transitive.

(iii) For any $r \in \mathbb{R}^n$ let $A \leqslant_C B$. Namely $B \subset A + C$. Then $\{r\} + B \subset \{r\} + A + C$. This implies $\{r\} + A \leqslant_C \{r\} + B$, i.e. the relation is compatible with vector addition.

(iv) For $\lambda > 0$ let $A \leqslant_C B$. Namely $B \subset A + C$. Then $\lambda B \subset \lambda(A + C)$ and since $C = \lambda C$ $\lambda B \subset \lambda A + C$. Hence the relation is compatible with scalar multiplication.

(v) To show the relation is not anti-symmetric we give an example. Let $C = \mathbb{R}^2_+ \subset \mathbb{R}^2$, $A := \{(0,0)\}$ and $B = C$. Since $A = \{(0,0)\} \subset \mathbb{R}^2_+ + C = B + C$ and $B = \mathbb{R}^2_+ \subset \{(0,0)\} + C = A + C$ then $A \leqslant_C B$ and $B \leqslant_C A$ but $A \neq B$. So the relation is not anti-symmetric. □

Lemma 4.6. *Let the relation "$\leqslant_K$" be as in Definition 4.4 for a total ordering cone K and A, B be two nonempty subsets of $\mathbb{R}^n$. Then either $A \leqslant_K B$ or $B \leqslant_K A$.*

Proof. Assume the contrary that $A \nleqslant_K B$ and $B \nleqslant_K A$ for some $A, B \subset \mathbb{R}^n$. Then there exist $\tilde{a} \in A$ such that $\tilde{a} \notin B + K$ and $\tilde{b} \in B$ such that $\tilde{b} \notin A + K$. Therefore $\tilde{a} \notin \{\tilde{b}\} + K$ and $\tilde{b} \notin \{\tilde{a}\} + K$. Namely $\tilde{a} \nleqslant_K \tilde{b}$ and $\tilde{b} \nleqslant_K \tilde{a}$. Since K is a total ordering cone this is a contradiction. □

Theorem 4.7. *Let X be a nonempty set, $C \subset \mathbb{R}^n$ be a cone with a compact base and nonempty interior $int(C)$ and $F : X \rightrightarrows \mathbb{R}^n$ be a C-closed, C-bounded set-valued map. If K is the total ordering cone which is obtained by cone C in (2) and $V_F : X \to \mathbb{R}^n$ is the K-minimal function of F, then $\{V_F(x)\} + K = F(x) + K$ for all $x \in X$.*

Proof. Since $V_F(x)$ is the minimal element of $F(x)$ with respect to the total ordering cone K then by Theorem 2.4 it is also strongly minimal element, i.e., $F(x) \subset \{V_F(x)\} + K$. Then $F(x) + K \subset (\{V_F(x)\} + K) + K = V_F(x) + K$. So,

$$F(x) + K \subset \{V_F(x)\} + K. \tag{6}$$

Since $V_F(x) \in F(x)$ then $\{V_F(x)\} \subset F(x)$. Hence

$$\{V_F(x)\} + K \subset F(x) + K. \tag{7}$$

From (6) and (7)

$$\{V_F(x)\} + K = F(x) + K. \qquad \square$$

Corollary 4.8. *Let X be a nonempty set and $C \subset \mathbb{R}^n$ be a cone with a compact base and with nonempty interior $int(C)$. If $F : X \rightrightarrows \mathbb{R}^n$ is a C-closed, C-bounded set-valued map, $V_F : X \to \mathbb{R}^n$ is the K-minimal function of F and K is a total ordering cone in (2). Then,*

$$F(x_1) \leqslant_K F(x_2) \quad \Leftrightarrow \quad V_F(x_1) \leqslant_K V_F(x_2)$$

for any $x_1, x_2 \in X$.

Proof. By Lemma 4.6 $V_F(x_1) + K = F(x_1) + K$ and $V_F(x_2) + K = F(x_2) + K$.

($\Rightarrow$) Let $F(x_1) \leqslant_K F(x_2)$, i.e., $F(x_2) \subset F(x_1) + K$ then

$$V_F(x_2) \in F(x_2) \subset F(x_1) + K = \{V_F(x_1)\} + K.$$

$V_F(x_2) \in \{V_F(x_1)\} + K$ implies $V_F(x_1) \leqslant_K V_F(x_2)$.

($\Leftarrow$) Let $V_F(x_1) \leqslant_K V_F(x_2)$, i.e., $V_F(x_2) \in \{V_F(x_1)\} + K$ then

$$F(x_2) \subset F(x_2) + K = \{V_F(x_2)\} + K \subset \{V_F(x_1)\} + K = F(x_1) + K.$$

$F(x_2) \subset F(x_1) + K$ means $F(x_1) \leqslant_K F(x_2)$. $\square$

Corollary 4.9. *Let X be a nonempty set and $C \subset \mathbb{R}^n$ be a cone with a compact base and with nonempty interior $int(C)$. If $F : X \rightrightarrows \mathbb{R}^n$ is C-closed, C-bounded set-valued map, $V_F : X \to \mathbb{R}^n$ is the K-minimal function of F and K is a total ordering cone in (2), then the solution of the set-valued optimization problem*

$$(SVP) \quad \begin{cases} \min F(x) \\ s.t.\ x \in X \end{cases} \tag{8}$$

with respect to the total order cone K is the same with the solution of the vector optimization problem

$$(VP) \quad \begin{cases} \min V_F(x) \\ s.t.\ x \in X. \end{cases} \tag{9}$$

Proof. We get this directly from Corollary 4.8. $\square$

The set-valued map and the K-minimal function derived from this set-valued map have many common properties. But, continuity is not one of them. The following example shows that the continuity of the set-valued map does not imply the continuity of the K-minimal function. The second example shows that the continuity of the K-minimal function does not imply the continuity of the set-valued map, either.

Example 4.10. Let $F : [0, 2\pi) \rightrightarrows \mathbb{R}^2$, $F(x) = [(0,0), (\cos x, \sin x)]$, $C = \mathbb{R}^2_+$ and the total ordering cone K is the cone we get by orthogonal vectors $r_1 = (1,1)$, $r_2 = (-1,1)$ that

$$K = \{(x,y) \in \mathbb{R}^2\colon x + y > 0\} \cup \{(x,y) \in \mathbb{R}^2\colon x < 0,\ y = -x\} \cup \{(0,0)\}.$$

Since $F(x)$ is a compact-valued map, it is C-bounded and C-closed. Moreover, it is continuous with respect to Hausdorff metric. The K-minimal function of $F(x)$ with respect to K is in the form of:

$$V_F(x) = \begin{cases} (\cos x, \sin x), & x \in \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right), \\ (0,0), & x \notin \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right). \end{cases} \tag{10}$$

And, it is obvious that $V_F(x)$ is not continuous at $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

Example 4.11. Let C and K be as in Example 4.10 and let the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as:

$$F(x) = \begin{cases} [(0,0),(1,2)], & x \in \mathbb{Q}, \\ [(0,0),(2,1)], & x \notin \mathbb{Q}. \end{cases} \tag{11}$$

It is obvious that $F(x)$ is not continuous at any point. But the K-minimal function of $F(x)$ is $V_F(x) = (0,0)$, for all $x \in \mathbb{R}$ and it is continuous.

5. Conclusions

In this study, set-valued optimization problem is converted into vector-valued optimization problem by using vectorization. This conversion is achieved without losing the vectorial structure of set-valued map. Then, we showed that the solution to the vector-optimization problem is the same as the solution to the set-valued optimization problem.

In addition, we introduced the Successive Weighted Sum Method, which is an advanced method based on the Weighted Sum Method. Vectorization presents a new perspective to set-valued optimization problems.

New methods can be introduced to improve vectorization.

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