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The generalization of total ordering cones and vectorization to separable Hilbert spaces

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ABSTRACT

The characterization of total ordering cones of $\mathbb{R}^n$ was given with some properties and optimality conditions in Küçük et al. (2011) [1]. In addition, total ordering cones were used to derive a vector valued function from a special class of set valued mappings in Küçük et al. (2012) [2]. In this study, we give a method for construction of a total ordering cone in a separable Hilbert space by using an orthogonal base. Moreover, we show that every total order can be represented by such a cone. The relationship between the notion of total ordering cone and the notion of vectorization of some set valued mappings are given and some results are obtained.

1. Introduction

The main purpose of vector optimization is to find the optimal elements of a set or optimal values of a vector valued function in a partially ordered space. In the same way, in the case of finding optimal sets in a family of sets or optimal values of a set valued mapping in a partially ordered space we get set valued optimization problem (one can see [3–7] for more studies on set-valued analysis and optimization). The choice of partial order in a vector optimization problem or set valued optimization problem is important. As the partial order determines which vector or set is better or optimal. Generally pointed, convex cones are used to define a partial order [8]. As known in this order some elements may not be comparable.

In [1], the question that under which condition a pointed convex cone defines a total order on a vector space was answered. Also a characterization of total orders was given with some properties and total orders were matched with orthogonal bases of $\mathbb{R}^n$. Moreover, optimality conditions of vector and set valued problems with respect to a total order were presented. By using these conditions Successive Weighted Sum Scalarization Method which was used not only for the problems with respect to a total ordering cone but also for the problems with respect to a cone with a compact base was obtained. (One can see [2] an example and the comparison of Successive Weighted Sum Method and Weighted Sum Method.) It was shown that, by using vectorization derived from set valued map, set-valued optimization problems can be represented as vector valued problems [9].

In this study, we extend the total ordering cones of $\mathbb{R}^n$ to separable real Hilbert spaces. We construct a total ordering cone for a real separable Hilbert space by using any orthogonal base of it.

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2. Mathematical preliminaries

In this section, we give some important properties and definitions which will be used in this article.

Proposition 2.1. *(See [1].) Let X be a vector space, C be a partial ordering cone in X and "$\leqslant_C$" be a partial order on X defined by*

$$a \leqslant_C b \quad \Leftrightarrow \quad b - a \in C.$$

Then

$$\text{“} \leqslant_C \text{” is a total order on } X \quad \text{if and only if} \quad C \cup (-C) = X.$$

3. Total ordering cones of separable real Hilbert spaces

Let $\{r_i\colon\ i \in \mathbb{N}^+\}$ be an orthogonal base of Separable Hilbert space X. Consider the sets $K_1 = \{x \in X\colon\ \langle r_1, x\rangle > 0\}$, $K_2 = \{x \in X\colon\ \langle r_1, x\rangle = 0,\ \langle r_2, x\rangle > 0\}$ and for any arbitrary $i \in \mathbb{N}^+$ $K_i = \{x \in X\colon\ \forall j < i,\ \langle r_j, x\rangle = 0,\ \langle r_i, x\rangle > 0\}$. Define the set

$$K = \left(\bigcup_{i \in \mathbb{N}^+} K_i\right) \cup \{0\}. \tag{1}$$

The following theorem shows that K is a total ordering cone in X derived from the base $\{r_i\colon\ i \in \mathbb{N}^+\}$ with respect to given order.

Theorem 3.1. *The set $K \subset X$ defined in (1) is a pointed convex cone with $K \cup (-K) = X$, i.e. K is a total ordering cone in X such that*

$$a \leqslant_K b \quad \Leftrightarrow \quad \exists i \in \mathbb{N}^+ \quad \text{such that} \quad \forall j < i, \quad \langle r_j, a\rangle = \langle r_j, b\rangle \quad \text{and} \quad \langle r_i, a\rangle < \langle r_i, b\rangle.$$

Proof.

i) First, we show that K is a cone. Let $k \in K$ and $\lambda \geqslant 0$.
If $k = 0_X$ or $\lambda = 0$ then $\lambda k = 0_X \in K$. So, let $k \neq 0_X$ and $\lambda > 0$. Then, there exists $i \in \mathbb{N}^+$ such that for each $j < i$ $\langle r_j, k\rangle = 0$ and $\langle r_i, k\rangle > 0$. So, for each $j < i$ $\langle r_j, \lambda k\rangle = 0$ and $\langle r_i, \lambda k\rangle > 0$. Therefore, $k \in K$. Hence, K is a cone.

ii) To show the pointedness of K, assume that there exists a nonzero vector $k \in K \cap (-K)$. Since, $k \in K$ there exists $i_1 \in \mathbb{N}^+$ such that for all $j < i_1$ $\langle r_j, k\rangle = 0$ and $\langle r_{i_1}, k\rangle > 0$. In the same way, since $k \in (-K)$, then there exists $i_2 \in \mathbb{N}^+$ such that for all $j < i_2$ $\langle r_j, k\rangle = 0$ and $\langle r_{i_2}, k\rangle < 0$, there are two cases for i_1 and i_2: $i_1 = i_2$ or $i_1 \neq i_2$.
a) If $i_1 = i_2$ then $\langle r_{i_1}, k\rangle > 0$ and $\langle r_{i_1}, k\rangle < 0$, which is a contradiction.
b) Let $i_1 < i_2$. Then for all $j < i_2$ $\langle r_j, k\rangle = 0$, which contradicts to $\langle r_{i_1}, k\rangle > 0$.
For the case $i_2 < i_1$, we get the same contradiction with b). Hence, $K \cap (-K) = \{0_X\}$. In other words, K is pointed.

iii) Now, we show the convexity of K. As, K is a cone it is enough to show that $K + K = K$.
Since $0_X \in K$,

$$K = 0_X + K \subset K + K. \tag{2}$$

To show the inverse inclusion, let $k \in K + K$. Then there are $k_1, k_2 \in K$ such that $k = k_1 + k_2$. If k_1 or k_2 is zero then $k \in K$ is obvious. So, we can assume that both k_1 and k_2 are nonzero. Then, there exists $i_1 \in \mathbb{N}^+$ such that for all $j < i$ $\langle r_j, k_1\rangle = 0$ and $\langle r_{i_1}, k_1\rangle > 0$ and there exists $i_2 \in \mathbb{N}^+$ such that for all $j < i_2$ $\langle r_j, k_2\rangle = 0$ and $\langle r_{i_2}, k_2\rangle > 0$. Let $i_1 \leqslant i_2$ Then

$$\langle r_j, k_1 + k_2\rangle = \langle r_j, k_1\rangle + \langle r_j, k_2\rangle = 0 + 0 = 0$$

for all $j < i_1$. Since, $\langle r_{i_1}, k_1\rangle > 0$ and $\langle r_{i_1}, k_2\rangle \geqslant 0$, we get

$$\langle r_{i_1}, k_1 + k_2\rangle = \langle r_{i_1}, k_1\rangle + \langle r_{i_1}, k_2\rangle > 0.$$

Then, we have $k_1 + k_2 = k \in K$. So, we obtain the desired inclusion

$$K + K \subset K. \tag{3}$$

By the inclusions (2) and (3), we have $K = K + K$. Hence, K is convex.

iv) Now, we can show that K is a total ordering cone. Since, we already showed that K is a partial ordering cone. By Proposition 2 in [1], it is enough to show that $K \cup (-K) = X$. The following inclusion is obvious

$$K \cup (-K) \subset X. \tag{4}$$

Let us show the converse inclusion. To do this let $r \in X$. If $r = 0_X$, then $r \in K \cup (-K)$. If $r \neq 0_X$, since $\{r_i\colon\ i \in \mathbb{N}^+\}$ is a base of X, then there exists $i \in \mathbb{N}^+$ such that $\langle r_j, r\rangle = 0$, for all $j < i$ and $\langle r_i, r\rangle \neq 0$. Because $\langle r_i, r\rangle \neq 0$, either $\langle r_i, r\rangle > 0$ or $\langle r_i, r\rangle < 0$.

If $\langle r_i, r\rangle > 0$ then $r \in K$.

If $\langle r_i, r\rangle < 0$ then $r \in -K$.

As a result, we obtain $r \in K \cup (-K)$. This implies

$$X \subset K \cup (-K). \tag{5}$$

By the inclusions (4) and (5), we get $X = K \cup (-K)$.

Moreover K defines a total order on X in the following way:

$$\begin{aligned} a \leqslant_K b \quad &\Leftrightarrow \quad b - a \in K \\ &\Leftrightarrow \quad \exists i \in \mathbb{N}^+ \quad \text{such that} \quad \langle r_j, b - a\rangle = 0, \quad \forall j < i \quad \text{and} \quad \langle r_i, b - a\rangle > 0 \\ &\Leftrightarrow \quad \exists i \in \mathbb{N}^+ \quad \text{such that} \quad \langle r_j, b\rangle = \langle r_j, a\rangle, \quad \forall j < i \quad \text{and} \quad \langle r_i, b\rangle > \langle r_i, a\rangle. \qquad \square \end{aligned}$$

Now, we can give some properties of total ordering cones in real separable Hilbert spaces.

Lemma 3.2. *Let K be a total ordering cone in a real separable Hilbert space X, then*

$$-K = (X \setminus K) \cup \{0_X\}.$$

Proof. Let $k \in -K$. If $k = 0_X$ then $k \in (X \setminus K) \cup \{0_X\}$. If $k \neq 0_X$ and $k \in K$ then we have $k \in K \cap (-K)$. But this contradicts with the pointedness of K. So, $k \in X \setminus K$ and

$$-K \subset (X \setminus K) \cup \{0_X\}. \tag{6}$$

To show the inverse inclusion, let $k \in (X \setminus K) \cup \{0_X\}$. If $k = 0_X$, then $k \in -K$. If $k \in X \setminus K$, since $X = K \cup (-K)$, then, $k \in -K$. So, we obtain

$$(X \setminus K) \cup \{0_X\} \subset -K. \tag{7}$$

By the inclusions (6) and (7), we have

$$-K = (X \setminus K) \cup \{0_X\}. \qquad \square$$

In Lemma 3.3, we show that if we reduce a total ordering cone to a subspace then we get a total ordering cone in that subspace.

Lemma 3.3. *In a real separable Hilbert space X, the intersection of a total ordering cone K and a subspace A of X is a total ordering cone in A.*

Proof. Let $K_A = K \cap A$. Since A and K are cones, K_A is a cone. Since A and K are convex, K_A is also convex. K is a pointed cone. Then, we have

$$K_A \cap (-K_A) = (A \cap K) \cap \big(A \cap (-K)\big) = A \cap \big(K \cap (-K)\big) = A \cap \{0\} = \{0_A\},$$

i.e. K_A is pointed. Hence, K_A is a pointed, convex cone.

Now, let us show K_A is a total ordering cone, i.e. $K_A \cup (-K_A) = A$.

$$A = A \cap X = A \cap \big(K \cup (-K)\big) = (A \cap K) \cup \big(A \cap (-K)\big) = K_A \cup (-K_A).$$

Hence, K_A is a pointed, convex cone with $K_A \cup (-K_A) = A$. Namely, K_A defines a total order on A. $\square$

The following lemma is a result of the separation of convex sets. (See Theorem 3.14 in [10].)

Lemma 3.4. *Let K be a total ordering cone in a real separable Hilbert space X. Then, there exists a vector $r \in X$ such that*

$$\big\{a \in X\colon\ \langle r, a\rangle > 0\big\} \subset K.$$

Proof. Since K and $(-K)$ are total ordering cones in X, they are convex. Besides, $int(K) \neq \emptyset$, K is pointed and $int(K) \cap (-K) = \emptyset$. By the separation of the convex sets there exist an $\ell \in X \setminus \{0_X\}$ and an $\alpha \in \mathbb{R}$ such that

$$\langle \ell, k_1 \rangle \leqslant \alpha \leqslant \langle \ell, k_2 \rangle \tag{8}$$

for all $k_1 \in K$ and for all $k_2 \in -K$. Moreover,

$$\langle \ell, k_1 \rangle < \alpha \tag{9}$$

for all $k_1 \in int(K)$.

Since K and $(-K)$ are cones, then $\alpha = 0$. If we choose $\ell = -r \in X$, then the inequality (8) turns into $\langle r, k_1 \rangle > 0$ for all $k_1 \in int(K)$.

Now we show that $\{a \in X\colon \langle r, a \rangle > 0\} \subset K$. Assume the contrary that $\langle r, a \rangle > 0$ for a vector $a \in X$ and $a \notin K$. By Lemma 3.2, $a \in -K$. But this contradicts with the inequality (8). Hence, $\{a \in X\colon \langle r, a \rangle > 0\} \subset K$. □

Theorem 3.5. *Let K be a total ordering cone in a real separable Hilbert space X. Then there exists a set $\{r_i\colon i \in \mathbb{N}^+\}$ such that $r_i \neq 0_X$, $\forall i \in \mathbb{N}^+$, $\langle r_j, r_i \rangle = 0$, $\forall j < i$ and*

$$K = \left(\bigcup_{i \in \mathbb{N}^+} \bigl\{ k \in X\colon \forall j < i,\ \langle k, r_j \rangle = 0,\ \langle k, r_i \rangle > 0 \bigr\} \right) \cup \{0_X\}. \tag{10}$$

Proof. By Lemma 3.4, there exists a vector $r_1 \in X$ such that

$$\bigl\{ k \in X\colon \langle r_1, k \rangle > 0 \bigr\} \subset K.$$

If we define the subspace $A_1 = \{a \in X\colon \langle r_1, a \rangle = 0\}$ then by Lemma 3.3 $K_{A_1} = K \cap A_1$ is a total ordering cone in A_1. By Lemma 3.4, there exists $r_2 \in A_1$ such that

$$\bigl\{ k \in X\colon \langle r_1, k \rangle = 0,\ \langle r_2, k \rangle > 0 \bigr\} \subset K_{A_1} \subset K.$$

Then we obtain the subspace $A_2 = \{a \in X\colon \langle r_1, a \rangle = 0,\ \langle r_2, a \rangle = 0\}$. $K_{A_2} = K \cap A_2$ is also a total ordering cone in A_2 and K_{A_2} contains an open half space

$$\bigl\{ k \in X\colon \langle r_1, k \rangle = 0,\ \langle r_2, k \rangle = 0,\ \langle r_3, k \rangle > 0 \bigr\} \subset K_{A_2} \subset K$$

and a subspace $A_3 = \{a \in X\colon \langle r_j, a \rangle = 0,\ \text{for all } j \leqslant 3\}$ which we get by using a nonzero vector $r_3 \in A_2$.

Continuing through this process, let $\{r_1, r_2, \ldots, r_n\}$ orthogonal set exists such that

$$\bigl\{ k \in X\colon \langle r_j, k \rangle = 0,\ \text{for all } j < n,\ \text{and } \langle r_n, k \rangle > 0 \bigr\} \subset K.$$

Then $A_n = \{a \in X\colon \langle r_j, a \rangle = 0,\ \text{for all } j \leqslant n\}$ is a subspace of X, $A_n \subset A_{n-1} = \{a \in X\colon \langle r_j, a \rangle = 0,\ \text{for all } j \leqslant n-1\}$, by Lemma 3.3 $K_{A_n} = K \cap A_n$ is a total ordering cone in A_n and by Lemma 3.4 there exists a nonzero vector r_{n+1} such that

$$\bigl\{ k \in X\colon \langle r_j, k \rangle = 0,\ \text{for all } j < n+1,\ \text{and } \langle r_{n+1}, k \rangle > 0 \bigr\} \subset K_{A_n} \subset K.$$

So, we obtain the orthogonal set $\{r_1, r_2, \ldots, r_{n+1}\}$ such that $A_{n+1} = \{a \in X\colon \langle r_j, a \rangle = 0,\ \text{for all } j \leqslant n+1\}$ is a subspace of X, $A_{n+1} \subset A_n = \{a \in X\colon \langle r_j, a \rangle = 0,\ \text{for all } j \leqslant n\}$, by Lemma 3.3 $K_{A_{n+1}} = K \cap A_{n+1}$ is a total ordering cone in A_{n+1} and by Lemma 3.4 there exists a nonzero vector r_{n+2} such that

$$\bigl\{ k \in X\colon \langle r_j, k \rangle = 0,\ \text{for all } j < n+2,\ \text{and } \langle r_{n+2}, k \rangle > 0 \bigr\} \subset K_{A_{n+1}} \subset K.$$

These implies:

$$\left(\bigcup_{i \in \mathbb{N}^+} \bigl\{ k \in X\colon \forall j < i,\ \langle k, r_j \rangle = 0,\ \langle k, r_i \rangle > 0 \bigr\} \right) \cup \{0_X\} \subset K. \tag{11}$$

It is obvious that $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ for the subspaces we constructed above. Then we have a reducing sequence of closed subspaces. So from Cantor's Theorem, the intersection of these subspaces consists of only one vector. As all subspaces contain 0_X, the intersection of this sequence is $\{0_X\}$. Namely,

$$\bigcap_{i \in \mathbb{N}^+} A_i = \{0_X\}.$$

$\{r_i\colon i \in \mathbb{N}^+\}$ is an orthogonal set and the orthogonal compliment of this set is $\{0_X\}$. So, $\{r_i\colon i \in \mathbb{N}^+\}$ is a countable base of the space. By Theorem 3.1, the union in the left hand side of (11) is a total ordering cone. To complete the proof it is enough to show that K is a subset of this union. Assume that a nonzero vector $k \in K$ is not an element of the union. Since the union is a total ordering cone then k belongs to negative of the union contained by $(-K)$. So, $k \in K \cap (-K)$ which contradicts with the pointedness of K. □

Definition 3.6. (See [10].) A nonempty convex subset B of a convex cone $C \neq \{0_X\}$ is called a base of C, if each $x \in C \setminus \{0_X\}$ has a unique representation of the form

$$x = \lambda b \quad \text{for some } \lambda > 0 \text{ and some } b \in B.$$

In addition, if B is compact then C is said to have a compact base.

Theorem 3.7. *Let C be a cone in real separable Hilbert space X. If C has a compact base then there exists a total ordering cone K such that*

$$C \setminus \{0_X\} \subset int(K).$$

Proof. Let the cone C have a compact base B. By Proposition 1.10 in [11] there exists an $r \in X$ such that $B = \{c \in C\colon \langle r, c\rangle = 1\}$. Then, we have $C \setminus \{0_X\} \subset \{a \in X\colon \langle r, a\rangle > 0\}$. If we choose $r_1 = r$ and construct the set $\{r_i\colon i \in \mathbb{N}^+ \setminus \{1\}\}$ as in the subspaces in the proof of Theorem 3.5, then the set $\{r_i\colon i \in \mathbb{N}^+\}$ is an orthogonal base of X and we obtain

$$\begin{aligned} C \setminus \{0_X\} &\subset \big\{a \in X\colon \langle r, a\rangle > 0\big\} \\ &\subset int\Bigg(\bigg(\bigcup_{i\in I}\big\{a \in X\colon \forall j < i,\ \langle r_j, a\rangle = 0,\ \langle r_i, a\rangle > 0\big\}\bigg) \cup \{0_X\}\Bigg) \\ &= int(K). \qquad \square \end{aligned}$$

Definition 3.8. Let C be a pointed, convex ordering cone in a real separable Hilbert space X, A be a nonempty subset of X for an element $\bar{x} \in A$ which satisfies $(\{\bar{x}\} - C) \cap A = \{\bar{x}\}$. $\bar{x}$ is called a minimal element of A with respect to the cone C. The set of all minimal elements A with respect to the cone C is denoted by $\min(A, C)$.

If $A \subset \{\bar{x}\} + C$ then $\bar{x}$ is called the strongly minimal of A with respect to the cone C.

Definition 3.9. Let C be a cone in a real separable Hilbert space X and A be a nonempty subset of X.

If $A + C$ is closed, then A is called C-closed set. If there exists $x \in X$ such that $A \subset \{x\} + C$ then A is called C-bounded set.

Theorem 3.10. *Let X be a real separable Hilbert space, $C \subset X$ be a cone with a compact base, $int(C) \neq \emptyset$ and $S \subset X$ be a C-closed and C-bounded set. Then there exists an $s \in S$ such that $\{s\} = \min(S, K)$.*

Proof. Since C has a compact base, there exist an orthogonal set $\{r_i\colon i \in \mathbb{N}^+\}$ and a total ordering cone K such that

$$K = \bigg(\bigcup_{i\in I}\big\{a \in X\colon \forall j < i,\ \langle r_j, a\rangle = 0,\ \langle r_i, a\rangle > 0\big\}\bigg) \cup \{0_X\}. \tag{12}$$

By Theorem 3.7,

$$C \subset \big\{r \in X\colon \langle r_1, r\rangle > 0\big\} \cup \{0_X\}$$

and $B := \{c \in C\colon \langle r_1, r\rangle = 1\}$ is a compact base of C. Because, S is C-bounded then there exists $x \in X$ such that

$$S \subset \{x\} + C.$$

Hence, $x \leqslant_C \tilde{x}$ for all $\tilde{x} \in S$. Since $r_1 \in C^\sharp$, $\langle r_1, \cdot\rangle$ is strictly increasing with respect to the cone C [10] then, we have $\langle r_1, x\rangle \leqslant \langle r_1, \tilde{x}\rangle$ for all $\tilde{x} \in S$. So, the set $\{\langle r_1, \tilde{x}\rangle\colon \tilde{x} \in S\}$ is bounded from below.

As the minimal elements of S are also the minimal elements of $S + C$, the set $\{\langle r_1, \tilde{x}\rangle\colon \tilde{x} \in S + C\}$ is also bounded from below. Because S is C-closed, $S + C$ is closed. This implies

$$a := \min\big\{\langle r_1, \tilde{x}\rangle\colon \tilde{x} \in S\big\}.$$

Let $b := \langle r_1, x\rangle$. It is obvious that $b \leqslant a$.

The set of minimal elements of the scalar problem

$$(SP_1) \quad \begin{cases} \min\langle r_1, \tilde{x}\rangle, \\ \text{s.t. } \tilde{x} \in S \end{cases} \tag{13}$$

is denoted by (Fig. 1)

$$S \cap \big((a - b)B + \{x\}\big).$$

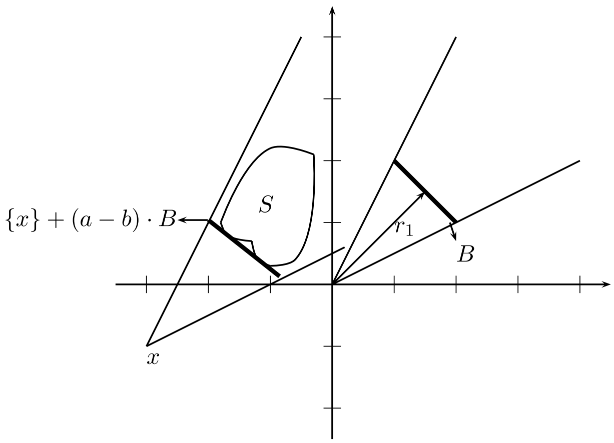

Fig. 1. The set of the minimal elements of the problem (SP_1).

Since, the minimal elements of S and $S+C$ are the same, we have the equation

$$(S+C)\cap\big((a-b)B+\{x\}\big)=S\cap\big((a-b)B+\{x\}\big).$$

As $S+C$ is closed, $((a-b)B+\{x\})$ is compact and the intersection of a closed and compact sets are compact, the set

$$A_1 := S\cap\big((a-b)B+\{x\}\big)$$

is compact.

The set of the minimal elements of the scalar problem

$$(SP_2)\quad \begin{cases}\min\langle r_2,\tilde{x}\rangle,\\ \text{s.t. } \tilde{x}\in A_1\end{cases} \tag{14}$$

is in the form of

$$A_2 := A_1\cap\big\{r\in X\colon\ \langle r_2,r\rangle=c_2\big\}$$

for a $c_2\in\mathbb{R}$. Since, A_1 is compact and the hyperplane $\{r\in X\colon\ \langle r_2,r\rangle=c\}$ is closed A_2 is a nonempty compact set.

Let the scalar problem $n-1$

$$(SP_{n-1})\quad \begin{cases}\min\langle r_{n-1},\tilde{y}\rangle,\\ \text{s.t. } \tilde{y}\in A_{n-2}\end{cases} \tag{15}$$

exists with a nonempty compact set A_{n-2}. Then the solution set of this problem is in the form of

$$A_{n-1} := A_{n-2}\cap\big\{r\in X\colon\ \langle r_{n-1},r\rangle=c_{n-1}\big\}$$

for a $c_{n-1}\in\mathbb{R}$. It is obvious that A_{n-1} is also a nonempty compact set. Hence we can define the scalar problem

$$(SP_n)\quad \begin{cases}\min\langle r_n,\tilde{y}\rangle,\\ \text{s.t. } \tilde{y}\in A_{n-1}\end{cases} \tag{16}$$

with a nonempty compact set A_{n-1}. We can get scalar problems (SP_i) for every $i\in\mathbb{N}^+\setminus\{1,2\}$. The sets of minimal elements of these problems are nonempty compact sets. Then we have a sequence A_n of nonempty compact sets. By Cantor's Theorem the intersection of these sets is a point. If we choose this point as s, then we have

$$\{s\}=\min(S,K). \qquad \square$$

Lemma 3.11. *Let X be a real separable Hilbert space, $C\subset X$ be a cone with a compact base such that $int(C)\neq\emptyset$ and $S_1,S_2\subset X$ be nonempty C-closed and C-bounded sets. Then*

i) *λS_1 is C-closed and C-bounded for all $\lambda>0$.*
ii) *S_1+S_2 is C-bounded.*

Proof.

i) By the definition, S_1 is C-closed equivalent to $S_1 + C$ is closed. Then,

$$\lambda(S_1 + C) = \lambda S_1 + \lambda C = \lambda S_1 + C$$

is closed for any $\lambda > 0$. Hence λS_1 is C-closed. Because, S_1 is C-bounded there exists a vector $x \in X$ such that $S_1 \subset \{x\} + C$. Then, we have

$$\lambda S_1 \subset \lambda\big(\{x\} + C\big) = \{\lambda x\} + C.$$

So λS_1 is C-bounded.

ii) Since S_1 is C-bounded, there exists a vector $x_1 \in X$ such that $S_1 \subset \{x_1\} + C$. In the same way, there exists $x_2 \in X$ such that $S_2 \subset \{x_2\} + C$. Then, we get

$$S_1 + S_2 \subset \big(\{x_1\} + C\big) + \big(\{x_2\} + C\big) = \{x_1 + x_2\} + C + C = \{x_1 + x_2\} + C.$$

This means, $S_1 + S_2$ is C-bounded. □

Theorem 3.12. *Let X be a real separable Hilbert space, $C \subset X$ be a cone with a compact base such that $int(C) \neq \emptyset$, $S_1, S_2 \subset X$ be nonempty C-closed and C-bounded sets and $\{s_1\} = \min(S_1, K)$, $\{s_2\} = \min(S_2, K)$ for a total ordering cone K such that $C \setminus \{0_X\} \subset int(K)$. Then*

i) $\{\lambda s_1\} = \min(\lambda S_1, K)$ *for all* $\lambda > 0$.

ii) *Additionally, if $S_1 + S_2$ is C-closed then $\{s_1 + s_2\} = \min(S_1 + S_2, K)$.*

Proof.

i) If s_1 is a minimal element of S_1 with respect to the cone K then by definition we have $(\{s_1\} - K) \cap S_1 = \{s_1\}$. So, we get $\lambda(\{s_1\} - K) \cap \lambda S_1 = \lambda\{s_1\}$ by multiplying both sides by a positive λ. Hence, we obtain $(\{\lambda s_1\} - K) \cap \lambda S_1 = \{\lambda s_1\}$, i.e. $\{\lambda s_1\} = \min(\lambda S_1, K)$.

ii) By the definition of minimality, we have $(\{s_1\} - K) \cap S_1 = \{s_1\}$ and $(\{s_2\} - K) \cap S_2 = \{s_2\}$. Then, it is obvious that $s_1 + s_2 \in S_1 + S_2$ and also $s_1 + s_2 \in \{s_1 + s_2\} - K$. So, we get $s_1 + s_2 \in (\{s_1 + s_2\} - K) \cap (S_1 + S_2)$. Let this intersection have another element $\tilde{s} = \tilde{s}_1 + \tilde{s}_2$ such that

$$\tilde{s}_1 \in S_1, \qquad \tilde{s}_2 \in S_2 \tag{17}$$

and

$$\tilde{s}_1 + \tilde{s}_2 \in \{s_1 + s_2\} - K. \tag{18}$$

Moreover, we have $\tilde{s}_1 \in s_1 + K$ and $\tilde{s}_2 \in s_2 + K$, because of (17), the minimality of s_1, s_2 and minimality is equivalent to strong minimality with respect to a total ordering cone. So, we obtain $\tilde{s}_1 + \tilde{s}_2 \in \{s_1 + s_2\} + K$ by addition of these results. This implies with (18) that $\tilde{s}_1 + \tilde{s}_2 - s_1 - s_2 \in K \cap (-K) = \{0_X\}$. Hence, we get $\tilde{s}_1 + \tilde{s}_2 = s_1 + s_2$. Since, we assumed that $\tilde{s} = \tilde{s}_1 + \tilde{s}_2$ as a different vector we have a contradiction and $(\{s_1 + s_2\} - K) \cap (S_1 + S_2) = \{s_1 + s_2\}$. Namely, $\{s_1 + s_2\} = \min(S_1 + S_2, K)$. □

We can get vectorization of a set-valued mapping as a result of Theorem 3.10.

Corollary 3.13. *Let X be a real separable Hilbert space, $C \subset X$ be an ordering cone with a compact base such that $int(C) \neq \emptyset$, Y be an arbitrary nonempty set and $F : Y \rightrightarrows X$ be a C-closed, C-bounded set-valued mapping. Then there exist a vector valued function $V_F : Y \to X$ and a total ordering cone K such that $\{V_F(y)\} = \min(F(y), K)$ for all $y \in Y$.*

Proof. For any $y \in Y$, $F(y)$ is C-closed and C-bounded. From Theorem 3.5, there exists $x \in F(y)$ such that $\{x\} = \min(F(y), K)$. If we choose $V_F(y) = x$ then we complete the proof. □

In terms of a total ordering cone, the concepts of minimality and strong minimality are equivalent [1]. So, we directly get Corollary 3.14.

Corollary 3.14. *Let X be a real separable Hilbert space, $C \subset X$ be an ordering cone with a compact base such that $int(C) \neq \emptyset$, Y be an arbitrary nonempty set and $F : Y \rightrightarrows X$ be a C-closed, C-bounded set-valued mapping, $V_F : Y \to X$ be vector-valued function and K be the total ordering cone we get in Corollary 3.13. Then, we have*

$$\big\{V_F(y)\big\} + K = F(y) + K \quad \textit{for all } y \in Y.$$

In [12], the ordering sets with cones was presented. Now, we can say that ordering set-valued mappings and ordering their vector-valued functions we get by vectorization are equivalent.

Corollary 3.15. *Let X be a real separable Hilbert space, $C \subset X$ be an ordering cone with a compact base such that $int(C) \neq \emptyset$, Y be an arbitrary nonempty set and $F : Y \rightrightarrows X$ be a C-closed, C-bounded set-valued mapping, $V_F : Y \to X$ be vector-valued function and K be the total ordering cone we get in Corollary* 3.13*. Then, we have*

$$V_F(y_1) \leqslant_K V_F(y_2) \quad \Leftrightarrow \quad F(y_1) \leqslant_K F(y_2) \quad \text{for all } y_1, y_2 \in Y.$$

The equivalence in Corollary 3.15 provide us solving a vector-valued optimization problem instead of solving a set-valued optimization problem.

Corollary 3.16. *Let X be a real separable Hilbert space, $C \subset X$ be an ordering cone with a compact base such that $int(C) \neq \emptyset$, Y be an arbitrary nonempty set and $F : Y \rightrightarrows X$ be a C-closed, C-bounded set-valued mapping, $V_F : Y \to X$ is vector-valued function and K is the total ordering cone we get in Corollary* 3.13*. Then, the solution of the vector-valued optimization problem*

$$(VP) \quad \begin{cases} \min & V_F(y), \\ s.t. & y \in Y \end{cases}$$

in terms of the total ordering cone K is also the solution of the set-valued optimization problem

$$(SVP) \quad \begin{cases} \min & F(y), \\ s.t. & y \in Y \end{cases}$$

in terms of the total ordering cone K.

4. Conclusion

In this article, we extended the characterization of total ordering cones of $\mathbb{R}^n$ which was given in [1] to real separable Hilbert spaces. By using this characterization, we presented vectorization of set-valued mappings in infinite dimensional spaces. The vectorization enables us to transfer the properties of vectors and vector valued functions to suitable sets and set-valued mappings in terms of ordering cones. So, we can obtain some optimality conditions for sets directly from optimality conditions for vectors.

Vectorization is given under the assumptions of cone-bounded and cone-closed sets or set-valued mappings. These assumptions are not rarely faced with when one studies set-valued optimization. Some properties of these class of sets are given for multiplication with a scalar and addition of two sets. Vectorization results under addition and scalar multiplication of sets are also studied.

Total ordering cones and vectorization of sets by using total ordering cones are novel methods in set-valued optimization. Future research or applications using them may contribute to new theorems and results on set-valued optimization in infinite dimensional spaces.

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