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On nonsingularity of a polytope of matrices

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Abstract

The nonsingularity problem of a polytope of real matrices and its relation to the (robust) stability problem is considered. This problem is investigated by using the Bernstein expansion of the determinant function. Here we adapt the known Bernstein algorithm for checking the positivity of a multivariate polynomial on a box to the nonsingularity problem. It is shown that for a family of Z-matrices the positive stability problem is equivalent to the nonsingularity if this family has a stable member. It is established that the stability of the convex hull of real matrices $A_1, A_2, \ldots, A_k$ is equivalent to the nonsingularity of the convex hull of matrices $A_1, A_2, \ldots, A_k, \mathrm{j}I$ if A_1 is stable.

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Keywords: Polytope of matrices; Nonsingularity; Stability; Multivariate polynomial; Bernstein expansion

1. Introduction

Let $\mathbb{R}^n$ be the set of real n vectors, $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) be the set of $n \times n$ real (complex) matrices. For $A_i \in \mathbb{R}^{n\times n}$ ($i = 1, 2, \ldots, k$) define the polytope (convex hull)

$$\mathscr{A} = \mathrm{conv}\{A_1, A_2, \ldots, A_k\} \tag{1}$$

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as the set of all real convex combinations. (In Section 4, the set of all real convex combinations of complex matrices will be considered and such a set will also be called the convex hull.) If all matrices in $\mathscr{A}$ are nonsingular then the family $\mathscr{A}$ is said to be nonsingular.

If all eigenvalues of a matrix A lie on the open right half plane then A is said to be positive stable. A is Hurwitz stable if $-A$ is positive stable. If all matrices in the family $\mathscr{A}$ are stable then the family is said to be (robustly) stable.

If the convex hull of matrices $A_1, A_2, \ldots, A_k$ is stable (nonsingular) then all the positive combinations are also stable (nonsingular).

Nonsingularity and stability problems for a family of matrices and their relationship were studied in many works (see [1–10]). These problems are NP-hard [11]. Probabilistic approach to the solution of these problems was studied in [12].

In this paper for the nonsingularity and the stability problems of the polytope (1) we use the Bernstein expansion of the determinant function. Here we adapt the known algorithm [13,14] for checking the positivity of a multivariate polynomial on a box to the nonsingularity problem of the polytope (1). We show that if there is a positive stable member in a compact path-connected family of Z-matrices then from nonsingularity follows positive stability (Proposition 3.2). We establish that if there is a stable member in the polytope (1) then stability of (1) is equivalent to nonsingularity of the family $\operatorname{conv}\{A_1, A_2, \ldots, A_k, \mathrm{j}I\}$, where I is the identity matrix and $\mathrm{j}^2 = -1$.

In [3,5], it is shown that the nonsingularity of an interval matrix family can be tested by calculating a finite number of determinants and the number of determinants that have to be tested is exponential in the dimension of the matrices. In [6], the following criterion for the nonsingularity of the family (1) is obtained: The family (1) is nonsingular if and only if for every nonzero $x \in \mathbb{R}^n$ there exists $y = y(x) \in \mathbb{R}^n$ such that $\langle A_i x, y\rangle < 0$ for all $i = 1, 2, \ldots, k$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product. The major disadvantage of this criterion is that for a given x the general choice rule of $y = y(x)$ is unknown (see Example 2.3 in Section 2).

For the case $k = 2$, a polynomial time solution for the Hurwitz stability of the family (1) is proposed in [1,2]. An exponential time algorithm for checking stability of a symmetric interval matrix family is given in [9]. In the paper [7], the authors consider the Hurwitz stability problem for the polytope of real matrices $\left\{A_0 + \sum_{i=1}^{k} q_k A_k : (q_1, \ldots, q_k) \in Q\right\}$, where Q is a box. Here the problem is reduced to the global minimization of an appropriate norm. For any given error tolerance $\varepsilon > 0$, it is shown its attainment via solution of a finite number of linear programs. However, the dependence of the number of linear programs on n and k is unknown. In [6,8] the following criterion for stability of the family (1) is obtained: The family (1) is Hurwitz stable if and only if for every nonzero vector $x \in \mathbb{C}^{n\times 1}$ there exists a positive definite (Hermitian) matrix $P = P(x)$ such that $\operatorname{Re} x^* P A_i x < 0$ for all $i = 1, 2, \ldots, k$, where x^* denotes the complex conjugate transpose of x. Here, again as in the case of nonsingularity there are serious difficulties in the applications of this criterion. In [4] using the guardian map concept the stability problem of a polynomial matrix family with one uncertainty parameter is considered. The transformation of the stability problem for the family $\mathscr{A}$ into the nonsingularity problem for the family $\{A \oplus A : A \in \mathscr{A}\}$ is considered in [5, Chapters 4 and 17]. Here $A \oplus A$ is the Kronecker sum. This transformation is not efficient from the computational point of view since $A \oplus A$ has dimension $n^2 \times n^2$.

Comparing our results in this paper with the above results, note that we consider a matrix polytope in the general form (1). Another essentiality of the our algorithm is that it is sufficiently fast (see Examples 2.2, 2.3, 4.4). Nevertheless, a serious drawback of using Bernstein expansion is its need of computing time and memory which grow exponentially with the number of variables.

2. Nonsingularity via Bernstein expansion

In this section by using determinant function we investigate the nonsingularity problem of the family (1). For this purpose we use the Bernstein expansion of the determinant function of the family (1).

In [13,14], the algorithm for estimating the range and checking for the positivity of a multivariate polynomial over a box is given. In this algorithm the expansion of a multivariate polynomial into Bernstein polynomials is used. Let us briefly describe this algorithm. Here we adopt essentially the notations from [14].

Let $L = (i_1, i_2, \ldots, i_m)$ be m-tuple of nonnegative integers and for $x = (x_1, x_2, \ldots, x_m)$

$$x^L = x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}.$$

For $N = (n_1, \ldots, n_m)$

$$L \leqslant N \Leftrightarrow 0 \leqslant i_k \leqslant n_k \quad (k = 1, 2, \ldots, m).$$

An m-variate polynomial $p(x)$ is defined as

$$p(x) = \sum_{L \leqslant N} a_L x^L \quad (x \in \mathbb{R}^m). \tag{2}$$

Here N is called the degree of the polynomial $p(x)$.

The ith Bernstein polynomial of degree d is defined as

$$b_{d,i}(x) = \binom{d}{i} x^i (1 - x)^{d-i}, \quad 0 \leqslant i \leqslant d.$$

In the multivariate case, the Lth Bernstein polynomial of degree N is defined by

$$B_{N,L}(x) = b_{n_1,i_1}(x_1) \cdots b_{n_m,i_m}(x_m) \quad (x \in \mathbb{R}^m). \tag{3}$$

The transformation of a polynomial from its power form (2) into its Bernstein form results in

$$p(x) = \sum_{L \leqslant N} p_L(U) B_{N,L}(x), \tag{4}$$

where the Bernstein coefficients $p_L(U)$ of p over the m-dimensional unit box $U = [0, 1] \times \cdots \times [0, 1]$ are given by

$$p_L(U) = \sum_{J \leqslant L} \frac{\binom{L}{J}}{\binom{N}{J}} a_J \quad (L \leqslant N). \tag{5}$$

Here $\binom{N}{L}$ is defined as the product $\binom{n_1}{i_1} \cdots \binom{n_m}{i_m}$. In [15] a difference table method for computing the Bernstein coefficients efficiently that avoids the binomial coefficients and product appearing in (5) is described.

Denote

$$\underline{m} = \min\{p(x) : x \in U\}, \quad \bar{m} = \max\{p(x) : x \in U\},$$
$$\alpha = \min\{p_L(U) : L \leqslant N\}, \quad \beta = \max\{p_L(U) : L \leqslant N\}.$$

Theorem 2.1 [13]. *The inequalities*

$$\alpha \leqslant \underline{m} \leqslant \bar{m} \leqslant \beta \tag{6}$$

are satisfied.

Theorem 2.1 gives the bounds for the range of the multivariate polynomial (2) over the unit box U. In order to obtain the Bernstein coefficients and bounds over an arbitrary box D, the box D should be affinely mapped onto U. As a result a new polynomial is being obtained and its Bernstein coefficients are the Bernstein coefficients of the initial polynomial $p(x)$ (2) over D.

In order to obtain convergent bounds for the range of the polynomial (2) over the box U, the box U should be divided into two boxes. If the division is continued and one calculates the minimal and maximal Bernstein coefficients in each subdivision step, the calculated bounds converge to the exact bounds (provided that the diameter of subboxes tends to zero). Note that by the sweep procedure the explicit transformation of the subboxes generated by sweeps back to U is avoided.

By Theorem 2.1, if $\alpha > 0$ then the polynomial $p(x)$ (2) is positive on U and if $\beta < 0$ then $p(x)$ is negative on U. If $\alpha \leqslant 0$, $\beta \geqslant 0$ then by the bisection in the chosen coordinate direction the box U is divided into two boxes and the new Bernstein coefficients for the new boxes can be calculated easily by using the Bernstein coefficients $p_L(U)$ (see [13,14]). In [14], the selection rule for the coordinate direction of division is suggested. This rule is based on the partial derivatives of a polynomial in Bernstein form (see (4)). A new box on which the inequality $\alpha > 0$ or $\beta < 0$ is satisfied should be eliminated, since our polynomial has constant sign on this box. A box, on which the inequality $\alpha > 0$ or $\beta < 0$ is not satisfied should be divided into two new boxes.

Summarizing the above, we note the followings:

(a) The positivity (or negativity) of a multivariate polynomial over a box D can be tested by this algorithm. For this purpose the box D should be mapped affinely into the unit box U (this map changes the initial polynomial also) and the Bernstein coefficients (5) should be calculated.
(b) If Bernstein coefficients have no constant sign, the box D should be divided into two boxes on which we proceed as before. It is important to note that by this procedure, the explicit transformation of the subboxes back to U is avoided.
(c) A subbox on which the Bernstein coefficients have constant sign should be eliminated, since our aim is to check positivity (or negativity) on the whole box D and the polynomial has definite sign on this subbox.
(d) If a multivariate polynomial is positive (negative) on the box D the algorithm gives an affirmative answer after a finite number of steps.

Let us now return to the problem of nonsingularity (1). The polytope (1) is equal to the set

$$\mathscr{A} = \left\{ \begin{aligned} &A = \lambda_1 A_1 + \cdots + \lambda_{k-1} A_{k-1} + (1 - \lambda_1 - \cdots - \lambda_{k-1}) A_k \\ &\quad : \lambda_1 \in [0,1], \ldots, \lambda_{k-1} \in [0,1],\ \lambda_1 + \cdots + \lambda_{k-1} \leqslant 1 \end{aligned} \right\}.$$

Denote

$$\Lambda = \{(\lambda_1, \ldots, \lambda_{k-1}) : \lambda_1 \in [0,1], \ldots, \lambda_{k-1} \in [0,1],\ \lambda_1 + \cdots + \lambda_{k-1} \leqslant 1\}, \tag{7}$$

$$f(\lambda_1, \ldots, \lambda_{k-1}) = \det(A), \tag{8}$$

where $(\lambda_1, \ldots, \lambda_{k-1}) \in \Lambda$, $A \in \mathscr{A}$.

By continuity the family $\mathscr{A}$ is nonsingular if and only if the determinant function $f(\lambda_1, \dots, \lambda_{k-1})$ (8) is positive or negative on Λ. This function is a multivariate polynomial while the set Λ (7) is not a box. Nevertheless, the above algorithm can be easily adapted to this problem. Indeed by (c) (see above) we eliminate a subbox $D = [\alpha_1, \beta_1] \times \cdots \times [\alpha_{k-1}, \beta_{k-1}]$ if the array of Bernstein coefficients has constant sign. Now, for our problem, if the inequality $\alpha_1 + \cdots + \alpha_{k-1} \geqslant 1$ is satisfied, the subbox D will be automatically eliminated since this subbox is remaining outside the set Λ.

Example 2.2. Consider $\mathscr{A} = \operatorname{conv}\{A_1, A_2, A_3\}$ where

$$A_1 = \begin{pmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 5 & 0 & -1 \\ -1 & 3 & -1 \\ -1 & -4 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 3 \end{pmatrix}.$$

We have

$$\Lambda = \{(\lambda_1, \lambda_2) : \lambda_1 \in [0, 1], \lambda_2 \in [0, 1],\ \lambda_1 + \lambda_2 \leqslant 1\},$$

$$\mathscr{A} = \{\lambda_1 A_1 + \lambda_2 A_2 + (1 - \lambda_1 - \lambda_2) A_3 :\ (\lambda_1, \lambda_2) \in \Lambda\}$$

and the determinant function of this family is

$$f(\lambda_1, \lambda_2) = 2 + 8\lambda_1 - 5\lambda_1^2 + 9\lambda_2 - 33\lambda_1\lambda_2 + 10\lambda_1^2\lambda_2 - 18\lambda_2^2 + 17\lambda_1\lambda_2^2 + 10\lambda_2^3.$$

The array of Bernstein coefficients (5) is

$$B(U) = \begin{pmatrix} 2 & 5 & 2 & 3 \\ 6 & \frac{7}{2} & -\frac{13}{6} & -1 \\ 5 & \frac{1}{3} & -\frac{14}{3} & 0 \end{pmatrix}$$

and has no constant sign. Therefore the bisection procedure must be applied to this problem. The algorithm reports after 0.01 s that the determinant function $f(\lambda_1, \lambda_2)$ is positive on the set Λ. It requires 15 bisection steps (Fig. 1). There are two type of rectangles in the figure. The first type consists of rectangles on which all Bernstein coefficients are positive (the total number of such rectangles is 10). The second type of rectangles satisfy the condition $\alpha_1 + \alpha_2 \geqslant 1$ and have as a side dotted line segments (the total number of such rectangles is 6). These rectangles are out of consideration. Consequently $\mathscr{A}$ is nonsingular.

Note that the determinant function is not positive on the whole box $[0, 1] \times [0, 1]$. For example, it is zero for $\lambda_1 = 1, \lambda_2 = 1$ and is negative for $\lambda_1 = 0.7, \lambda_2 = 0.7$.

Example 2.3. Consider $\mathscr{A} = \operatorname{conv}\{A_1, A_2, A_3\}$, where

$$A_1 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

This example is taken from [6]. There the nonsingularity of this polytope is proved by choosing an appropriate map $y(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. Here we prove the nonsingularity of $\mathscr{A}$ by the Bernstein expansion. We have

$$f(\lambda_1, \lambda_2) = -1 - \lambda_1 + \lambda_1\lambda_2 + \lambda_2^2 - \lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - \lambda_2^3.$$

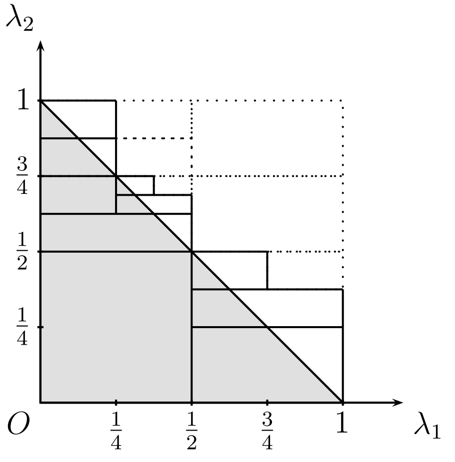

Fig. 1. Bisection of rectangles.

The array of Bernstein coefficients is

$$B(U) = \begin{pmatrix} -1 & -1 & -2/3 & -1 \\ -3/2 & -4/3 & -7/6 & -2 \\ -2 & -2 & -7/3 & -4 \end{pmatrix}$$

and all coefficients are negative. Hence $f(\lambda_1, \lambda_2) < 0$ on $[0, 1] \times [0, 1]$ and $\mathscr{A}$ is nonsingular.

Remark 2.4. If there exist two subboxes D_1 and D_2 such that all Bernstein coefficients on D_1 are positive and all Bernstein coefficients on D_2 are negative then the process should be terminated. In this case the family $\mathscr{A}$ is singular by continuity.

3. *Z*-matrices: positive stability is equivalent to nonsingularity

A real $n \times n$ matrix $A = [a_{ij}]$ is said to be a Z-matrix if $a_{ij} \leqslant 0$ for all $i \neq j$.

Theorem 3.1 [16, p. 114]. *Let $A \in \mathbb{R}^{n\times n}$ be Z-matrix. Then A has at least one real eigenvalue and is positively stable if and only if every real eigenvalue of A is positive.*

Stability problems of Z-matrices are studied in [16,17].

Proposition 3.2. *Let $\mathscr{A} \subset \mathbb{R}^{n\times n}$ be a compact path-connected family of Z-matrices. Assume that the family $\mathscr{A}$ has at least one positive stable matrix A_1. If the family $\mathscr{A}$ is nonsingular then it is positive stable.*

Proof. Suppose the contrary that the family $\mathscr{A}$ is not positive stable and assume that A_2 is not positive stable. Consider the continuous path $f(t) : [0, 1] \to \mathscr{A}$ connecting A_1 and A_2, i.e.

$$f(0) = A_1, \quad f(1) = A_2, \quad f(t) \in \mathscr{A} \quad (t \in [0, 1]).$$

Consider the motion of the eigenvalues of $f(t)$ starting at $t = 0$. By the continuity theorem of eigenvalues [5, p. 52] there exist continuous functions $s_i : [0, 1] \to \mathbb{C}$ $(i = 1, 2, \dots, n)$ such that $s_1(t), s_2(t), \dots, s_n(t)$ are the eigenvalues of $f(t)$. We set

$$\mathbb{C}^- = \{z \in \mathbb{C} : \operatorname{Re} z \leqslant 0\},$$

$$K = \{i : i \in \{1, 2, \dots, n\} \text{ and there exists } t \in [0, 1] \text{ such that } s_i(t) \in \mathbb{C}^-\}.$$

$\mathbb{C}^-$ is the closed left-half plane and $i \in K$ if and only if the curve $s_i(t)$ intersects the imaginary axis. K is non-empty since A_2 is not positive stable. For $i \in K$ define $T_i = \{t \in [0, 1] : s_i(t) \in \mathbb{C}^-\}$. From continuity of $s_i(t)$ and closedness of $\mathbb{C}^-$ it follows that T_i is compact. Let

$$t_i = \min T_i \quad (i \in K), \quad \tilde{t} = \min\{t_i : i \in K\}.$$

It is obvious that $s_i(t_i)$ belongs to the boundary of $\mathbb{C}^-$, i.e. the imaginary axis. $\tilde{t}$ is the minimal value of t such that at least one of the eigenvalues of $f(\tilde{t})$ lies on the imaginary axis and the matrix $f(\tau)$ is positive stable for all $\tau < \tilde{t}$. The matrix $f(\tilde{t})$ is not positive stable.

Let λ be an arbitrary real eigenvalue of the matrix $f(\tilde{t})$. It is evident that $\lambda \geqslant 0$. If $\lambda = 0$ then $f(\tilde{t})$ is singular which contradicts the nonsingularity assumption. If $\lambda > 0$ then $f(\tilde{t})$ is positive stable by Theorem 3.1. This contradiction shows that the family $\mathscr{A}$ is positive stable. □

Note that a similar result is true for positive definiteness of symmetric interval matrices, i.e. a symmetric interval matrix family is positive definite if and only if it contains at least one positive definite matrix and is nonsingular (see [10]).

Example 3.3. Consider the family $\mathscr{A}$ as in Example 2.2. This family consists of Z-matrices and it is nonsingular. Since A_1 is positive stable then the family $\mathscr{A}$ is positive stable by Proposition 3.2.

4. Stability of an arbitrary real polytope

Let the polytope (1) be given, where $A_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2, \dots, k)$. In this section we investigate the stability of the polytope (1).

Proposition 4.1. *Let the polytope* (1) *be given and* $B \in \mathbb{C}^{n\times n}$. *Then the following equality is true*

$$\{(1 - \lambda)A + \lambda B : \lambda \in [0, 1],\ A \in \mathscr{A}\} = \operatorname{conv}\{A_1, \dots, A_k, B\}. \tag{9}$$

Here

$$\operatorname{conv}\{A_1, \dots, A_k, B\} = \left\{\lambda_1 A_1 + \cdots + \lambda_k A_k + \lambda_{k+1} B : \sum_{i=1}^{k+1} \lambda_i = 1,\ \lambda_i \geqslant 0 \quad (i = 1, 2, \dots, k + 1)\right\}.$$

The proof is immediate.

Theorem 4.2. *Let the polytope* (1) *be given and* A_1 *is positive* (*Hurwitz*) *stable. Then the following are equivalent.*

(i) *All matrices in* $\mathscr{A}$ *are positively* (*Hurwitz*) *stable.*
(ii) *The polytope* $\operatorname{conv}\{A_1, \dots, A_k, jI\}$ *is nonsingular where* I *is the identity matrix.*

Proof. (i) $\Rightarrow$ (ii). From (i) it follows that for all $A \in \mathscr{A}$ and $\omega \leqslant 0$

$$\det(A - \mathrm{j}\omega I) \neq 0. \tag{10}$$

Let $\omega = -\frac{\lambda}{1-\lambda}$, where $\lambda \in [0, 1)$. Then from (10) it follows that:

$$\det((1-\lambda)A + \lambda \mathrm{j} I) \neq 0. \tag{11}$$

The relation (11) is also true for $\lambda = 1$. By Proposition 4.1 condition (ii) follows from (11).

(ii) $\Rightarrow$ (i). If (ii) is satisfied then by Proposition 4.1 the relation (11) and consequently (10) is true. Since the polytope $\mathscr{A}$ consists of real matrices it follows that $\mathscr{A}$ has no pure imaginary eigenvalue. On the other hand $\mathscr{A}$ has at least one stable member. Then (i) is true. Indeed, if $\mathscr{A}$ is not stable then there exists an unstable matrix in $\mathscr{A}$. Since $\mathscr{A}$ has at least one stable member, by continuity and connectedness there exists $A_* \in \mathscr{A}$ which has pure imaginary eigenvalues. This contradiction shows that (i) is true. $\square$

Remark 4.3. The condition (ii) in Theorem 4.2 can be replaced by the following condition: (ii′) The polytope $\mathrm{conv}\{A_1, \ldots, A_k, -jI\}$ is nonsingular.

Denote

$$\Lambda = \{(\lambda_1, \ldots, \lambda_k) : \lambda_1 \in [0, 1], \ldots, \lambda_k \in [0, 1], \lambda_1 + \cdots + \lambda_k \leqslant 1\},$$
$$F(\lambda_1, \ldots, \lambda_k) = \det(A)$$

where $(\lambda_1, \ldots, \lambda_k) \in \Lambda$, $A = \lambda_1 A_1 + \cdots + \lambda_k A_k + (1 - \lambda_1 - \cdots - \lambda_k) jI$.

The condition (ii) in Theorem 4.2 says that $F(\lambda_1, \ldots, \lambda_k) \neq 0$ for all $(\lambda_1, \ldots, \lambda_k) \in \Lambda$. This also can be tested by the Bernstein expansion. Indeed an arbitrary matrix A from the family $\mathrm{conv}\{A_1, \ldots, A_k, \mathrm{j}I\}$ can be written as

$$A = A_R + \mathrm{j}A_I$$

where $A_R = \lambda_1 A_1 + \cdots + \lambda_k A_k$, $A_I = (1 - \lambda_1 - \cdots - \lambda_k)I$. Consider a nonzero vector $x \in \mathbb{C}^{n\times 1}$, $x = x_R + \mathrm{j}x_I$ and the equation

$$(A_R + \mathrm{j}A_I)(x_R + \mathrm{j}x_I) = 0.$$

This equation can be written as

$$\begin{pmatrix} A_R & A_I \\ -A_I & A_R \end{pmatrix} \begin{pmatrix} x_R \\ -x_I \end{pmatrix} = 0.$$

This homogenous linear equation has a nonzero solution if and only if

$$\det \begin{pmatrix} A_R & A_I \\ -A_I & A_R \end{pmatrix} \neq 0. \tag{12}$$

This determinant is equal to $\det\left(A_R^2 + A_I^2\right)$ and is nonnegative. (If A, B, C, D are $n \times n$ matrices and $AB = BA$ then by the Schur identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(DA - CB),$$

see [18, p. 11]). The calculation gives a multivariate polynomial whose degree is doubled in comparison with (8).

An alternative way to investigate the equality $\det(\lambda_1 A_1 + \cdots + \lambda_{k+1} \mathrm{j} I) = 0$ on $\{\lambda = (\lambda_1, \ldots, \lambda_{k+1}) : \lambda_1 + \cdots + \lambda_{k+1} = 1, \lambda_i \in [0, 1], (i = 1, \ldots, k+1)\}$ is the use of a system of polynomial equations. Indeed assume that $\det(\lambda_1 A_1 + \cdots + \lambda_{k+1} \mathrm{j} I) = f(\lambda) + \mathrm{j} g(\lambda)$. Then we obtain the following system of polynomial equations over the $(k+1)$-dimensional unit box:

$$\begin{aligned} &f(\lambda) = 0, \\ &g(\lambda) = 0, \\ &\lambda_1 + \cdots + \lambda_{k+1} - 1 = 0. \end{aligned} \tag{13}$$

The existence problem of the solution of (13) can be investigated by the Bernstein expansion, domain-splitting and eliminations. A subbox $[\alpha_1, \beta_1] \times \cdots \times [\alpha_{k+1}, \beta_{k+1}]$ on which $f(\lambda) \neq 0$ or $g(\lambda) \neq 0$ or $\alpha_1 + \cdots + \alpha_{k+1} > 1$ or $\beta_1 + \cdots + \beta_{k+1} < 1$ should be eliminated. For the remaining subboxes the existence test provided by Miranda's theorem can be applied. This test provides a generalization of the fact that if a univariate continuous function f has a sign change at the end points of an interval then this interval contains a zero of f (for details see [19]).

Example 4.4. Let $\mathscr{A} = \mathrm{conv}\{A_1, A_2, A_3\}$ where

$$A_1 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

and A_1 is Hurwitz stable. The determinant function

$$f(\lambda_1, \lambda_2, \lambda_3) = \det(A_R^2 + A_I^2)$$

is a sextic polynomial.

In the table below, we give sweep directions, the minimal and maximal Bernstein coefficients, and the eliminated subboxes. Observe that it is not encountered a box $D = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_3, \beta_3]$ with $\alpha_1 + \alpha_2 + \alpha_3 \geqslant 1$ in the eliminations. The algorithm reports after 0.6 s that the determinant function $\det\left(A_R^2 + A_I^2\right)$ is positive on the box $U = [0,1] \times [0,1] \times [0,1]$ and by Theorem 4.2 the family $\mathscr{A}$ is Hurwitz stable.

Subboxes	Minimal Bernstein coefficients	Maximal Bernstein coefficients	Sweep directions and eliminated subboxes
$[0,1] \times [0,1] \times [0,1]$	$-5/18$	4325	Divide first interval
$\left[0, \frac{1}{2}\right] \times [0,1] \times [0,1]$	$-1/6$	69673/64	Divide second interval
$\left[\frac{1}{2}, 1\right] \times [0,1] \times [0,1]$	17/64	4325	Eliminate
$\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times [0,1]$	$-17/180$	1117/4	Divide third interval
$\left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right] \times [0,1]$	1/16	69673/64	Eliminate
$\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$	$-27/800$	4049/64	Divide first interval
$\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]$	5/48	1117/4	Eliminate
$\left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$	$-1531/115200$	63225/4096	Divide second interval
$\left[\frac{1}{4}, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$	959/61440	4049/64	Eliminate
$\left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{2}\right]$	5971/368640	4	Eliminate
$\left[0, \frac{1}{4}\right] \times \left[\frac{1}{4}, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$	19/1440	63225/4096	Eliminate

The solution of this example as a solution of the system of polynomial equations (13) is investigated also. The sweep directions are chosen as the directions in which the box edge lengths are larger. After 190 sweeps in 19 s the algorithm gives an affirmative answer.

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