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Research Article

Seiberg-Witten Like Equations on Pseudo-Riemannian Spin^c Manifolds with $G^*_{2(2)}$ Structure

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We consider 7-dimensional pseudo-Riemannian spin^c manifolds with structure group $G^*_{2(2)}$. On such manifolds, the space of 2-forms splits orthogonally into components $\Lambda^2 M = \Lambda^2_7 \oplus \Lambda^2_{14}$. We define self-duality of a 2-form by considering the part Λ^2_7 as the bundle of self-dual 2-forms. We express the spinor bundle and the Dirac operator and write down Seiberg-Witten like equations on such manifolds. Finally we get explicit forms of these equations on $\mathbb{R}^{4,3}$ and give some solutions.

1. Introduction

The Seiberg-Witten theory, introduced by Witten in [1], became one of the most important tools to understand the topology of smooth 4-manifolds. The Seiberg-Witten theory is based on the solution space of two equations which are called the Seiberg-Witten equations. The first one of the Seiberg-Witten equations is Dirac equation and the second one is known as curvature equation [2]. The first equation is the harmonicity condition of spinor fields; that is, the spinor field belongs to the kernel of the Dirac operator. The second equation couples the self-dual part of the curvature 2-form with a spinor field. There exist various generalizations of Seiberg-Witten equations to higher dimensional Riemannian manifolds [3–6]. All of these generalizations are done for the manifolds which have special structure groups. Also Seiberg-Witten like equations are studied over 4-dimensional Lorentzian spin^c manifolds [7] and 4-dimensional pseudo-Riemannian manifolds with neutral signature [8].

Parallel spinors on pseudo-Riemannian spin^c manifolds are studied by Ikemakhen [9]. In the present work, we consider 7-dimensional manifolds with structure group $G^*_{2(2)}$. In order to define spinors and Dirac operator, the manifold M must have a spin^c-structure. We assume that 7-dimensional pseudo-Riemannian manifold M with signature $(-,-,-,-,+,+,+)$ has spin^c-structure. On the other hand, to write down curvature equation, we need a self-duality notion of a 2-form on such manifolds. In 4 dimensions, self-duality concept of 2-forms is well known. The bundle of 2-forms $\Lambda^2(M)$ decomposes into two parts on this manifold [10]. Then we will define self-duality of a 2-form on a 7-manifold with structure group $G^*_{2(2)}$ by using decomposition of 2-forms on this manifold.

2. Manifolds with Structure Group $G^*_{2(2)}$

The exceptional Lie group G_2, automorphism group of octonions, is well known. There is another similar Lie group $G^*_{2(2)}$ which is automorphism group of split octonions [11]. On $\mathbb{R}^7$, we consider the metric

$$g_{4,3}(x,y) = -x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 + x_5y_5 + x_6y_6 + x_7y_7, \quad (1)$$

where $x = (x_1, x_2, \dots, x_7)$ and $y = (y_1, y_2, \dots, y_7) \in \mathbb{R}^7$. From now on, we denote the pair $(\mathbb{R}^7, g_{4,3})$ by $\mathbb{R}^{4,3}$. The isometry group of this space is

$$O(4,3) = \left\{A \in GL(7,\mathbb{R}) : g_{4,3}(A(x), A(y)) = g_{4,3}(x,y),\ \forall x, y \in \mathbb{R}^7\right\}. \quad (2)$$

The special orthogonal subgroup of $O(4,3)$ is

$$\mathrm{SO}(4,3) = \{A \in O(4,3) : \det A = 1\}. \quad (3)$$

The group $G^*_{2(2)}$ is the subgroup of $\mathrm{SO}(4,3)$, preserving the following 3-form:

$$\varphi_0 = -e^{127} - e^{135} + e^{146} + e^{236} + e^{245} - e^{347} + e^{567}, \quad (4)$$

where $\{e^1, \dots, e^7\}$ is the dual base of the standard basis $\{e_1, \dots, e_7\}$ of $\mathbb{R}^{4,3}$, with the notation $e^{ijk} = e^i \wedge e^j \wedge e^k$ and with the metric $g_{4,3} = (-1,-1,-1,-1,1,1,1)$; that is,

$$G^*_{2(2)} = \{A \in GL(7,\mathbb{R}) : A^*\varphi_0 = \varphi_0\}, \quad (5)$$

where φ_0 is called the fundamental 3-form on $\mathbb{R}^{4,3}$ [10, 11]. The space of 2-forms $\Lambda^2\mathbb{R}^7$ decomposes into two parts $\Lambda^2\mathbb{R}^7 = \Lambda^2_7\mathbb{R}^7 \oplus \Lambda^2_{14}\mathbb{R}^7$, where

$$\Lambda^2_7\mathbb{R}^7 = \{\alpha \in \Lambda^2\mathbb{R}^7 : \star(\varphi_0 \wedge \alpha) = 2\alpha\},$$
$$\Lambda^2_{14}\mathbb{R}^7 = \{\alpha \in \Lambda^2\mathbb{R}^7 : \star(\varphi_0 \wedge \alpha) = -\alpha\}. \quad (6)$$

A semi-Riemannian 7-manifold M with the metric of signature $(-,-,-,-,+,+,+)$ is called a $G^*_{2(2)}$ manifold if its structure group reduces to the Lie group $G^*_{2(2)}$; equivalently, there exists a nowhere vanishing 3-form on M whose local expression is of the form φ_0. Such a form is called a $G^*_{2(2)}$ structure on M [12]. If the structure group of M is the group $G^*_{2(2)}$ then the bundle of 2-forms $\Lambda^2(M)$ decomposes into two parts similar to $\Lambda^2\mathbb{R}^7$ and we denote it by $\Lambda^2(M) = \Lambda^2_7(M) \oplus \Lambda^2_{14}(M)$ [10].

It is known that square of the Hodge $*$ operator on 2-forms over 4-dimensional Riemannian manifolds is identity and ± 1 are eigenvalues of the Hodge $*$ operator. The elements of eigenspace of 1 are called self-dual 2-forms and the others are called anti-self-dual forms. But this situation does not generalize to higher dimensional manifolds directly. Self-duality of 2-form has been studied on some higher dimensions [3, 13]. In this work, we need self-duality concept of 2-forms on 7-dimensional manifolds with structure group $G^*_{2(2)}$.

Now we define a duality operator over bundle of 2-form $\Lambda^2(M)$ as

$$T_\varphi : \Lambda^2(M) \longrightarrow \Lambda^2(M),$$
$$T_\varphi(\alpha) := \star(\varphi \wedge \alpha). \quad (7)$$

The eigenvalues of this map are 2 and -1. Note that the subbundle $\Lambda^2_7(M)$ corresponds to the eigenvalue 2 and the subbundle $\Lambda^2_{14}(M)$ corresponds to the eigenvalue -1. Let α be a 2-form over M. If α belongs to $\Lambda^2_7(M)$, then we call α a self-dual 2-form. If α belongs to $\Lambda^2_{14}(M)$, then we call α an anti-self-dual 2-form. Because of decomposition of 2-forms on M, any 2-form α on M can be written uniquely as

$$\alpha = \alpha^+ + \alpha^-, \quad (8)$$

where $\alpha^+ \in \Lambda^2_7(M)$ and $\alpha^- \in \Lambda^2_{14}(M)$. Similar to the 4-dimensional case, we say that α^+ is self-dual part of α and α^- is anti-self-dual part of α.

3. Spinor Bundles over $G^*_{2(2)}$ Manifolds

It is known that the group $\mathrm{SO}(4,3)$ has two connected components. The connected component to the identity of $\mathrm{SO}(4,3)$ is denoted by $\mathrm{SO}_+(4,3)$. In this work we deal with the group $\mathrm{SO}_+(4,3)$. The covering space of $\mathrm{SO}(4,3)$ is the group $\mathrm{Spin}(4,3)$ which lies in Clifford algebra $\mathrm{Cl}_{4,3} = \mathrm{Cl}(\mathbb{R}^7, -g_{4,3}) \subset \mathbb{C}l_{4,3}$ and we denoted the connected component of $1 \in \mathrm{Spin}(4,3)$ by $\mathrm{Spin}_+(4,3)$. There is a covering map $\lambda : \mathrm{Spin}_+(4,3) \to \mathrm{SO}_+(4,3)$ which is a $2:1$ group homomorphism given by $\lambda(g)(x) = g \cdot x \cdot g^{-1}$ for $x \in \mathbb{R}^{4,3}$, $g \in \mathrm{Spin}_+(4,3)$ [10, 11, 14].

One can define another group which lies in the complex Clifford algebra $\mathbb{C}l(\mathbb{R}^{4,3}) \cong \mathbb{C}l_7$ by

$$\mathrm{Spin}^c_+(4,3) := \frac{(\mathrm{Spin}_+(4,3) \times S^1)}{\mathbb{Z}_2}, \quad (9)$$

where the elements of $\mathrm{Spin}^c_+(4,3)$ are the equivalence classes $[g,z]$ of pair $(g,z) \in \mathrm{Spin}_+(4,3) \times S^1$, under the equivalence relation $(g,z) \sim (-g,-z)$ [9]. There exist two exact sequences as

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_+(4,3) \xrightarrow{\lambda} \mathrm{SO}_+(4,3) \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c_+(4,3) \xrightarrow{\xi} \mathrm{SO}_+(4,3) \times S^1 \longrightarrow 1, \quad (10)$$

where $\xi([g,z]) = (\lambda(g), z^2)$.

Let $\{e_1, \dots, e_7\}$ be an orthonormal basis of $\mathbb{R}^{4,3}$; then the Lie algebras of $\mathrm{Spin}(4,3)$ and $\mathrm{Spin}^c(4,3)$ are

$$\mathrm{spin}(4,3) = \{e_i e_j : 1 \le i, \ j \le 7\},$$
$$\mathrm{spin}^c(4,3) = \mathrm{spin}(4,3) \oplus i\mathbb{R}, \quad (11)$$

respectively. The derivative of $\xi : \mathrm{Spin}^c_+(4,3) \to \mathrm{SO}_+(4,3) \times S^1$ is obtained as

$$\xi_*(e_i e_j, ir) = (\lambda_*(e_i e_j), ir) = (2E_{ij}, 2ir), \quad (12)$$

where E_{ij} is the 8×8-matrix whose (i,j)-entry is 1, (j,i)-entry is -1, and the other entries are zero [9]. Since the Clifford algebra $\mathbb{C}l_7$ is isomorphic to the algebra $\mathbb{C}(8) \oplus \mathbb{C}(8)$, we can project this isomorphism onto the first component. Hence, we get spinor representation:

$$\kappa : \mathbb{C}l_7 \longrightarrow \mathbb{C}(8) \cong \mathrm{End}(\mathbb{C}^8). \quad (13)$$

By restricting κ to the group $\mathrm{Spin}^c_+(4,3)$ we get

$$\kappa|_{\mathrm{Spin}^c_+(4,3)} : \mathrm{Spin}^c_+(4,3) \longrightarrow \mathrm{Aut}(\mathbb{C}^8) \quad (14)$$

and $\kappa|_{\mathrm{Spin}^c_+(4,3)}$ is called spinor representation of the group $\mathrm{Spin}^c_+(4,3)$; shortly we denote it by κ. The elements of $\mathbb{C}^8$ are called spinors and the complex vector space $\mathbb{C}^8$ is called the spinor space and it is denoted by $\Delta_{4,3}$. By using spinor representation, the Clifford multiplication of vectors with spinors is defined by

$$X \cdot \psi := \kappa(X)(\psi), \quad (15)$$

where $X \in \mathbb{R}^{4,3}$ and $\psi \in \Delta_{4,3}$. The spinor space has a nondegenerate indefinite Hermitian inner product as

$$\langle \psi_1, \psi_2 \rangle_{\Delta_{4,3}} := i^{4(4-1)/2} \langle \kappa(e_1 e_2 e_3 e_4) \psi_1, \psi_2 \rangle, \quad (16)$$

where $\langle z, w \rangle = \sum_{i=1}^{8} z_i \overline{w}_i$ is the standard Hermitian inner product on $\mathbb{C}^8$ for $z = (z_1, \ldots, z_8)$, $w = (w_1, \ldots, w_8) \in \mathbb{C}^8$. The new inner product $\langle\ ,\ \rangle_{\Delta_{4,3}}$ is invariant with respect to the group $\mathrm{spin}_+^c(4,3)$ and satisfies the following property:

$$\langle \kappa(Z) \psi_1, \psi_2 \rangle_{\Delta_{4,3}} = -\langle \psi_1, \kappa(Z) \psi_2 \rangle_{\Delta_{4,3}}, \quad (17)$$

where $Z \in \mathbb{R}^{4,3}$ and $\psi_1, \psi_2 \in \Delta_{4,3}$. In this work, we use the following spinor representation κ:

$$\begin{aligned}
\kappa(e_1) &= \varepsilon \otimes \varepsilon \otimes \delta, \\
\kappa(e_2) &= -\delta \otimes \delta \otimes \tau, \\
\kappa(e_3) &= -\delta \otimes I \otimes \delta, \\
\kappa(e_4) &= \delta \otimes \tau \otimes \tau, \\
\kappa(e_5) &= -I \otimes \varepsilon \otimes \tau, \\
\kappa(e_6) &= -\tau \otimes \varepsilon \otimes \delta, \\
\kappa(e_7) &= I \otimes I \otimes \varepsilon,
\end{aligned} \quad (18)$$

where

$$\begin{aligned}
I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\delta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\tau &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\varepsilon &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \quad (19)$$

Now, we recall the main definitions concerning spin^c-structure and the spinor bundle. Let M be a 7-dimensional pseudo-Riemannian manifold with structure group $G_{2(2)}^*$. Then, there is an open covering $\{U_\alpha\}_{\alpha \in A}$ of M and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G_{2(2)}^* \subset \mathrm{SO}_+(4,3)$ for TM.

If there exists another collection of transition functions

$$\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Spin}_+^c(4,3) \quad (20)$$

such that the following diagram commutes

$$\begin{array}{ccc}
 & & \mathrm{Spin}_+^c(4,3) \\
 & \overset{\widetilde{g}_{\alpha\beta}}{\nearrow} & \downarrow \xi \\
U_\alpha \cap U_\beta & \xrightarrow[g_{\alpha\beta}]{} & \mathrm{SO}_+(4,3)
\end{array} \quad (21)$$

(i.e., $\xi \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ and the cocycle condition $\widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma} = \widetilde{g}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ is satisfied), then M is called a spin^c manifold. Then one can construct a principal $\mathrm{Spin}_+^c(4,3)$-bundle $P_{\mathrm{Spin}_+^c(4,3)}$ on M and a bundle map $\Lambda : P_{\mathrm{Spin}_+^c(4,3)} \to P_{\mathrm{SO}_+(4,3)}$.

Let $(P_{\mathrm{Spin}_+^c(4,3)}, \Lambda)$ be a spin^c-structure on M. We can construct an associated complex vector bundle:

$$S = P_{\mathrm{Spin}_+^c(4,3)} \times_\kappa \Delta_{4,3}, \quad (22)$$

where $\kappa : \mathrm{Spin}_+^c(4,3) \to \mathrm{Aut}(\Delta_{4,3})$ is the spinor representation of $\mathrm{Spin}_+^c(4,3)$. This complex vector bundle is called spinor bundle for a given spin^c-structure on M and sections of S are called spinor fields. The Clifford multiplication given by (15) can be extended to a bundle map:

$$\mu : TM \otimes S \longrightarrow S. \quad (23)$$

Parallel spinors on the spinor bundle S are studied in [9].

Since M is a pseudo-Riemannian spin^c manifold, then by using the map

$$\begin{aligned}
\ell : \mathrm{Spin}_+^c(4,3) &\longrightarrow S^1, \\
\ell([g,z]) &= z^2,
\end{aligned} \quad (24)$$

we can get an associated principal S^1-bundle:

$$P_{S^1} = P_{\mathrm{Spin}_+^c(4,3)} \times_\ell S^1. \quad (25)$$

Also, the map ℓ induces a bundle map:

$$L : P_{\mathrm{Spin}_+^c(4,3)} \longrightarrow P_{S^1}. \quad (26)$$

Now, fix a connection 1-form $A : TP_{S^1} \to i\mathbb{R}$ over the principal $U(1)$-bundle P_{S^1}. Let ∇ be the Levi-Civita covariant derivative associated with the metric $g_{4,3}$ which determines an $\mathrm{so}(4,3)$-valued connection 1-form ω on the principal bundle $P_{\mathrm{SO}_+(4,3)}$. The connection 1-form ω can be written locally

$$\omega = \sum_{i<j} \omega_{ij} E_{ij}, \quad (27)$$

where $\{e_1, e_2, \ldots, e_7\}$ is a local orthonormal frame on open set $U \subset M$ and $\omega_{ij} = g_{4,3}(\nabla e_i, e_j)$. By using the connection 1-form A and ω, one can obtain a connection 1-form on the principal bundle $P_{\mathrm{SO}_+(4,3)} \widetilde{\times} P_{S^1}$ (the fibre product bundle):

$$\omega \times A : T\left(P_{\mathrm{SO}_+(4,3)} \widetilde{\times} P_{S^1}\right) \longrightarrow \mathrm{SO}_+(4,3) \times i\mathbb{R}. \quad (28)$$

The connection $\omega \times A$ can be lift to a connection 1-form Z^A on the principal bundle $P_{\mathrm{SO}_+^c(4,3)}$ via the 2-fold covering map:

$$\pi := (\Lambda, L) : P_{\mathrm{Spin}_+^c(4,3)} \longrightarrow P_{\mathrm{SO}_+(4,3)} \widetilde{\times} P_{S^1} \quad (29)$$

and the following commutative diagram.

$$\begin{array}{ccc}
T\left(P_{\mathrm{Spin}_+^c(4,3)}\right) & \xrightarrow{Z^A} & \mathrm{Lie}\left(\mathrm{Spin}_+^c(4,3)\right) \cong \mathrm{spin}(4,3) \oplus i\mathbb{R} \\
d\pi \downarrow & & \downarrow \xi_* \\
T\left(P_{\mathrm{SO}_+(4,3)} \widetilde{\times} P_{S^1}\right) & \xrightarrow{\omega \times A} & \mathrm{SO}(4,3) \oplus i\mathbb{R}
\end{array} \quad (30)$$

One can obtain a covariant derivative operator ∇^A on the spinor bundle S by using the connection 1-form Z^A. The local form of the covariant derivative ∇^A is

$$\nabla^A \Psi = d\Psi + \frac{1}{2}\sum_{i<j} \varepsilon_i \varepsilon_j \omega_{ij} \kappa \left(e_i e_j\right) \Psi + \frac{1}{2} A\Psi, \tag{31}$$

where $\{e_1, \ldots, e_7\}$ is a orthonormal frame on open set $U \subset M$. We note that some authors use the term $A\Psi$ instead of $(1/2)A\Psi$ in the local formula of $\nabla^A\Psi$. The covariant derivative ∇^A is compatible with the metric $\langle\ ,\ \rangle_{\Delta_{4,3}}$

$$X \langle \psi_1, \psi_2 \rangle_{\Delta_{4,3}} = \left\langle \nabla^A_X \psi_1, \psi_2 \right\rangle_{\Delta_{4,3}} + \left\langle \psi_1, \nabla^A_X \psi_2 \right\rangle_{\Delta_{4,3}} \tag{32}$$

and the Clifford multiplication

$$\nabla^A_X \left(Y \cdot \psi\right) = Y \cdot \nabla^A_X \psi + \left(\nabla_X Y\right) \cdot \psi, \tag{33}$$

where ψ, ψ_1, ψ_2 are spinor fields and sections of S, X, and Y are vector fields on M. We can define the Dirac operator D_A as the following composition:

$$\begin{aligned} D_A &:= \mu \circ \nabla^A : \Gamma(S) \xrightarrow{\nabla^A} \Gamma\left(TM^* \otimes S\right) \overset{g_{4,3}}{\simeq} (TM \otimes S) \\ &\xrightarrow{\mu} \Gamma(S), \end{aligned} \tag{34}$$

which can be written locally as

$$D_A(\psi) = \sum_{i=1}^{7} \varepsilon_i \kappa\left(e_i\right) \nabla^A_{e_i}(\psi), \tag{35}$$

where $\{e_1, e_2, \ldots, e_7\}$ is any oriented local orthonormal frame of TM.

4. Seiberg-Witten Like Equations on $G^*_{2(2)}$ Manifolds

Let M be a spin^c manifold with structure group $G^*_{2(2)}$. Fix a spin^c-structure and a connection A in the principal $U(1)$-bundle P_{S^1} associated with the spin^c-structure. Note that the curvature F_A of the connection A is $i\mathbb{R}$-valued 2-form. The curvature 2-form F_A on the P_{S^1} determines an $i\mathbb{R}$-valued 2-form on M uniquely (see [15]) and we denote it again by F_A.

We can define a map

$$\sigma(\psi)(X, Y) = \langle X \cdot Y \cdot \psi, \psi \rangle_{\Delta_{4,3}} + g_{4,3}(X, Y) |\psi|^2, \tag{36}$$

where $X, Y \in \Gamma(TM)$. Note that the map $\sigma(\psi)$ satisfies the following properties:

$$\begin{aligned} \sigma(\psi)(X, Y) &= -\sigma(\psi)(Y, X), \\ \overline{\sigma(\psi)(X, Y)} &= -\sigma(\psi)(X, Y). \end{aligned} \tag{37}$$

Hence, the map σ associates an $i\mathbb{R}$-valued 2-form with each spinor field $\psi \in \Gamma(S)$, so we can write

$$\sigma : \Gamma(S) \longrightarrow \Omega^2(M, i\mathbb{R}). \tag{38}$$

In local frame $\{e_1, e_2, \ldots, e_7\}$ on $U \subset M$, the map σ can be expressed as

$$\sigma(\psi) = -\frac{1}{4}\sum_{i<j} \left\langle \kappa\left(e_i e_j\right)\psi, \psi \right\rangle_{\Delta_{4,3}} e_i \wedge e_j. \tag{39}$$

Now we are ready to express the Seiberg-Witten equations. Let M be a spin^c manifold with structure group $G^*_{2(2)}$. Fix a $\mathrm{Spin}^c_+(4,3)$ structure and take a connection 1-form A on the principal bundle P_{S^1} and a spinor field $\psi \in \Gamma(S)$. We write the Seiberg-Witten like equations as

$$\begin{aligned} D_A \psi &= 0, \\ F^+_A &= -\frac{1}{4}\sigma(\psi)^+, \end{aligned} \tag{40}$$

where F^+_A is the self-dual part of the curvature F_A and $\sigma(\psi)^+$ is the self-dual part of the 2-form $\sigma(\psi)$ corresponding to the spinor $\psi \in \Gamma(S)$.

5. Seiberg-Witten Like Equations on $\mathbb{R}^{4,3}$

Let us consider these equations on the flat space $M = \mathbb{R}^{4,3}$ with the $G^*_{2(2)}$ structure given by φ_0. We use the standard orthonormal frame $\{e_1, e_2, \ldots, e_7\}$ on $M = \mathbb{R}^{4,3}$ and the spinor representation in (18). The spin^c connection ∇^A on $\mathbb{R}^{4,3}$ is given by

$$\nabla^A_j \Psi = \frac{\partial \Psi}{\partial x_j} + A_j \Psi, \tag{41}$$

where $A_j : \mathbb{R}^{4,3} \to i\mathbb{R}$ and $\Psi : \mathbb{R}^{4,3} \to \Delta_{4,3}$ are smooth maps. Then, the associated connection on the line bundle $L_\Gamma = \mathbb{R}^{4,3} \times \mathbb{C}$ is the connection 1-form

$$A = \sum_{i=1}^{7} A_i dx_i \in \Omega^1\left(\mathbb{R}^{4,3}, i\mathbb{R}\right) \tag{42}$$

and its curvature 2-form is given by

$$F_A = dA = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2\left(\mathbb{R}^{4,3}, i\mathbb{R}\right), \tag{43}$$

where $F_{ij} = \partial A_j/\partial x_i - \partial A_i/\partial x_j$ for $i, j = 1, \ldots, 7$. Now we can write the Dirac operator D_A on $\mathbb{R}^{4,3}$ with respect to a given spin^c-structure and spin^c-connection ∇^A.

We denote the dual basis of $\{e_1, e_2, \dots, e_7\}$ by $\{e^1, e^2, \dots, e^7\}$. Now one can give a frame for the space of self-dual 2-forms on $\mathbb{R}^{4,3}$ as

$$\begin{aligned} f_1 &= e^1 \wedge e^2 + e^3 \wedge e^4 - e^5 \wedge e^6, \\ f_2 &= e^1 \wedge e^3 - e^2 \wedge e^4 - e^6 \wedge e^7, \\ f_3 &= e^1 \wedge e^4 + e^2 \wedge e^3 - e^5 \wedge e^7, \\ f_4 &= e^1 \wedge e^5 - e^2 \wedge e^6 - e^4 \wedge e^7, \\ f_5 &= e^1 \wedge e^6 + e^2 \wedge e^5 - e^3 \wedge e^7, \\ f_6 &= e^1 \wedge e^7 + e^3 \wedge e^6 + e^4 \wedge e^5, \\ f_7 &= e^2 \wedge e^7 + e^3 \wedge e^5 - e^4 \wedge e^6. \end{aligned} \tag{44}$$

Let F_A be the curvature form of the $i\mathbb{R}$-valued connection 1-form A and let F_A^+ be its self-dual part. Then,

$$\begin{aligned} F_A^+ &= \sum_{i=1}^{7} \langle F_A, f_i \rangle \frac{f_i}{|f_i|^2} = \frac{1}{3} \{(F_{12} + F_{34} - F_{56}) f_1 \\ &\quad + (F_{13} - F_{24} - F_{67}) f_2 + (F_{14} + F_{23} - F_{57}) f_3 \\ &\quad + (F_{15} - F_{26} - F_{47}) f_4 + (F_{16} + F_{25} - F_{37}) f_5 \\ &\quad + (F_{17} + F_{36} + F_{45}) f_6 + (F_{27} + F_{35} - F_{46}) f_7 \}. \end{aligned} \tag{45}$$

Now we calculate the 2-form $\sigma(\psi)^+$, for a spinor $\psi \in S$. Then $\sigma(\psi)$ can be written in the following way:

$$\sigma(\psi) = \sum_{i<j} \langle e_i e_j \psi, \psi \rangle e^i \wedge e^j. \tag{46}$$

The projection onto the subspace $\Lambda_7^2(\mathbb{R}^{4,3}, i\mathbb{R})$ is given by

$$\sigma(\psi)^+ = \sum_{i=1}^{7} \langle \sigma(\psi), f_i \rangle \frac{f_i}{|f_i|^2}. \tag{47}$$

If $\sigma(\psi)^+$ is calculated explicitly, then we obtain the following identity:

$$\begin{aligned} 3\sigma(\psi)^+ &= \{-3\psi_2\overline{\psi}_1 + 3\psi_1\overline{\psi}_2 + \psi_4\overline{\psi}_3 - \psi_3\overline{\psi}_4 - \psi_6\overline{\psi}_5 \\ &\quad + \psi_5\overline{\psi}_6 - \psi_8\overline{\psi}_7 + \psi_7\overline{\psi}_8\} f_1 + \{3\psi_3\overline{\psi}_1 + \psi_4\overline{\psi}_2 \\ &\quad - 3\psi_1\overline{\psi}_3 - \psi_2\overline{\psi}_4 + \psi_7\overline{\psi}_5 - \psi_8\overline{\psi}_6 - \psi_5\overline{\psi}_7 + \psi_6\overline{\psi}_8\} \\ &\quad \cdot f_2 + \{-3\psi_4\overline{\psi}_1 + \psi_3\overline{\psi}_2 - \psi_2\overline{\psi}_3 + 3\psi_1\overline{\psi}_4 + \psi_8\overline{\psi}_5 \\ &\quad + \psi_7\overline{\psi}_6 - \psi_6\overline{\psi}_7 - \psi_5\overline{\psi}_8\} f_3 + \{-3\psi_6\overline{\psi}_1 + \psi_5\overline{\psi}_2 \\ &\quad + \psi_8\overline{\psi}_3 + \psi_7\overline{\psi}_4 - \psi_2\overline{\psi}_5 + 3\psi_1\overline{\psi}_6 - \psi_4\overline{\psi}_7 - \psi_3\overline{\psi}_8\} \\ &\quad \cdot f_4 + \{-3\psi_5\overline{\psi}_1 - \psi_6\overline{\psi}_2 - \psi_7\overline{\psi}_3 + \psi_8\overline{\psi}_4 + 3\psi_1\overline{\psi}_5 \\ &\quad + \psi_2\overline{\psi}_6 + \psi_3\overline{\psi}_7 - \psi_4\overline{\psi}_8\} f_5 + \{-3\psi_7\overline{\psi}_1 - \psi_8\overline{\psi}_2 \\ &\quad + \psi_5\overline{\psi}_3 - \psi_6\overline{\psi}_4 - \psi_3\overline{\psi}_5 + \psi_4\overline{\psi}_6 + 3\psi_1\overline{\psi}_7 + \psi_2\overline{\psi}_8\} \\ &\quad \cdot f_6 + \{-3\psi_8\overline{\psi}_1 + \psi_7\overline{\psi}_2 - \psi_6\overline{\psi}_3 - \psi_5\overline{\psi}_4 + \psi_4\overline{\psi}_5 \\ &\quad + \psi_3\overline{\psi}_6 - \psi_2\overline{\psi}_7 + 3\psi_1\overline{\psi}_8\} f_7. \end{aligned} \tag{48}$$

Hence, the curvature equation can be written explicitly as

$$\begin{aligned} F_{12} + F_{34} - F_{56} &= \frac{1}{4} \{3\psi_2\overline{\psi}_1 - 3\psi_1\overline{\psi}_2 - \psi_4\overline{\psi}_3 + \psi_3\overline{\psi}_4 \\ &\quad + \psi_6\overline{\psi}_5 - \psi_5\overline{\psi}_6 + \psi_8\overline{\psi}_7 - \psi_7\overline{\psi}_8\}, \\ F_{13} - F_{24} - F_{67} &= \frac{1}{4} \{-3\psi_3\overline{\psi}_1 - \psi_4\overline{\psi}_2 + 3\psi_1\overline{\psi}_3 \\ &\quad + \psi_2\overline{\psi}_4 - \psi_7\overline{\psi}_5 + \psi_8\overline{\psi}_6 + \psi_5\overline{\psi}_7 - \psi_6\overline{\psi}_8\}, \\ F_{14} + F_{23} - F_{57} &= \frac{1}{4} \{3\psi_4\overline{\psi}_1 - \psi_3\overline{\psi}_2 + \psi_2\overline{\psi}_3 - 3\psi_1\overline{\psi}_4 \\ &\quad - \psi_8\overline{\psi}_5 - \psi_7\overline{\psi}_6 + \psi_6\overline{\psi}_7 + \psi_5\overline{\psi}_8\}, \\ F_{15} - F_{26} - F_{47} &= \frac{1}{4} \{-3\psi_6\overline{\psi}_1 + \psi_5\overline{\psi}_2 + \psi_8\overline{\psi}_3 + \psi_7\overline{\psi}_4 \\ &\quad - \psi_2\overline{\psi}_5 + 3\psi_1\overline{\psi}_6 - \psi_4\overline{\psi}_7 - \psi_3\overline{\psi}_8\}, \\ F_{16} + F_{25} - F_{37} &= \frac{1}{4} \{-3\psi_5\overline{\psi}_1 - \psi_6\overline{\psi}_2 - \psi_7\overline{\psi}_3 + \psi_8\overline{\psi}_4 \\ &\quad + 3\psi_1\overline{\psi}_5 + \psi_2\overline{\psi}_6 + \psi_3\overline{\psi}_7 - \psi_4\overline{\psi}_8\}, \\ F_{17} + F_{36} + F_{45} &= \frac{1}{4} \{-3\psi_7\overline{\psi}_1 - \psi_8\overline{\psi}_2 + \psi_5\overline{\psi}_3 - \psi_6\overline{\psi}_4 \\ &\quad - \psi_3\overline{\psi}_5 + \psi_4\overline{\psi}_6 + 3\psi_1\overline{\psi}_7 + \psi_2\overline{\psi}_8\}, \\ F_{27} + F_{35} - F_{46} &= \frac{1}{4} \{-3\psi_8\overline{\psi}_1 + \psi_7\overline{\psi}_2 - \psi_6\overline{\psi}_3 - \psi_5\overline{\psi}_4 \\ &\quad + \psi_4\overline{\psi}_5 + \psi_3\overline{\psi}_6 - \psi_2\overline{\psi}_7 + 3\psi_1\overline{\psi}_8\}. \end{aligned} \tag{49}$$

Dirac equation $D_A\Psi = 0$ can be expressed as follows:

$$\begin{aligned} &\frac{\partial\psi_8}{\partial x_1} - \frac{\partial\psi_7}{\partial x_2} - \frac{\partial\psi_6}{\partial x_3} + \frac{\partial\psi_5}{\partial x_4} - \frac{\partial\psi_3}{\partial x_5} - \frac{\partial\psi_4}{\partial x_6} + \frac{\partial\psi_2}{\partial x_7} \\ &\quad = -A_1\psi_8 + A_2\psi_7 + A_3\psi_6 - A_4\psi_5 + A_5\psi_3 \\ &\qquad + A_6\psi_4 - A_7\psi_2, \\ &\frac{\partial\psi_7}{\partial x_1} + \frac{\partial\psi_8}{\partial x_2} - \frac{\partial\psi_5}{\partial x_3} - \frac{\partial\psi_6}{\partial x_4} + \frac{\partial\psi_4}{\partial x_5} - \frac{\partial\psi_3}{\partial x_6} - \frac{\partial\psi_1}{\partial x_7} \\ &\quad = -A_1\psi_7 - A_2\psi_8 + A_3\psi_5 + A_4\psi_6 - A_5\psi_4 \\ &\qquad + A_6\psi_3 + A_7\psi_1, \\ &-\frac{\partial\psi_6}{\partial x_1} - \frac{\partial\psi_5}{\partial x_2} - \frac{\partial\psi_8}{\partial x_3} - \frac{\partial\psi_7}{\partial x_4} + \frac{\partial\psi_1}{\partial x_5} + \frac{\partial\psi_2}{\partial x_6} + \frac{\partial\psi_4}{\partial x_7} \\ &\quad = A_1\psi_6 + A_2\psi_5 + A_3\psi_8 + A_4\psi_7 - A_5\psi_1 - A_6\psi_2 \\ &\qquad - A_7\psi_4, \\ &-\frac{\partial\psi_5}{\partial x_1} + \frac{\partial\psi_6}{\partial x_2} - \frac{\partial\psi_7}{\partial x_3} + \frac{\partial\psi_8}{\partial x_4} - \frac{\partial\psi_2}{\partial x_5} + \frac{\partial\psi_1}{\partial x_6} + \frac{\partial\psi_3}{\partial x_7} \\ &\quad = A_1\psi_5 - A_2\psi_6 + A_3\psi_7 - A_4\psi_8 + A_5\psi_2 - A_6\psi_1 \\ &\qquad - A_7\psi_3, \end{aligned}$$

$$-\frac{\partial\psi_4}{\partial x_1}-\frac{\partial\psi_3}{\partial x_2}-\frac{\partial\psi_2}{\partial x_3}+\frac{\partial\psi_1}{\partial x_4}-\frac{\partial\psi_7}{\partial x_5}+\frac{\partial\psi_8}{\partial x_6}+\frac{\partial\psi_6}{\partial x_7}$$
$$=A_1\psi_4+A_2\psi_3+A_3\psi_2-A_4\psi_1+A_5\psi_7-A_6\psi_8-A_7\psi_6,$$
$$-\frac{\partial\psi_3}{\partial x_1}+\frac{\partial\psi_4}{\partial x_2}-\frac{\partial\psi_1}{\partial x_3}-\frac{\partial\psi_2}{\partial x_4}+\frac{\partial\psi_8}{\partial x_5}+\frac{\partial\psi_7}{\partial x_6}-\frac{\partial\psi_5}{\partial x_7}$$
$$=A_1\psi_3-A_2\psi_4+A_3\psi_1+A_4\psi_2-A_5\psi_8-A_6\psi_7+A_7\psi_5,$$
$$\frac{\partial\psi_2}{\partial x_1}-\frac{\partial\psi_1}{\partial x_2}-\frac{\partial\psi_4}{\partial x_3}-\frac{\partial\psi_3}{\partial x_4}+\frac{\partial\psi_5}{\partial x_5}-\frac{\partial\psi_6}{\partial x_6}+\frac{\partial\psi_8}{\partial x_7}$$
$$=-A_1\psi_2+A_2\psi_1+A_3\psi_4+A_4\psi_3-A_5\psi_5+A_6\psi_6-A_7\psi_8,$$
$$\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}-\frac{\partial\psi_3}{\partial x_3}+\frac{\partial\psi_4}{\partial x_4}-\frac{\partial\psi_6}{\partial x_5}-\frac{\partial\psi_5}{\partial x_6}-\frac{\partial\psi_7}{\partial x_7}$$
$$=-A_1\psi_1-A_2\psi_2+A_3\psi_3-A_4\psi_4+A_5\psi_6+A_6\psi_5+A_7\psi_7. \tag{50}$$

These equations admit nontrivial solutions. For example, direct calculation shows that the spinor field

$$\psi=\left(0,0,\psi_3,i\psi_3,\psi_3,i\psi_3,0,0\right) \tag{51}$$

with $\psi_3(x_1,x_2,\ldots,x_7) = e^{-(i/2)x_1^2x_2}$ and the connection 1-form

$$A\left(x_1,x_2,\ldots,x_7\right)=\left(ix_1x_2\right)dx_1+\left(\frac{i}{2}x_1^2\right)dx_2 \tag{52}$$

satisfy the above equations.

Now we consider the space

$$\mathscr{C}=\mathscr{A}\times\Gamma\left(S\right), \tag{53}$$

where $\mathscr{A}$ is the space of connection 1-forms on the principle bundle P_{S^1} and $\Gamma(S)$ is the space of spinor fields. The space $\mathscr{C}$ is called the configuration space. There is an action of the gauge group $\mathscr{G} := \mathrm{Map}(X,S^1)$ on the configuration space by

$$u\cdot\left(A,\psi\right):=\left(A+u^{-1}du,u^{-1}\psi\right), \tag{54}$$

where $u\in\mathscr{G}$ and $(A,\psi)\in\mathscr{C}$. The action of the gauge group enjoys the following equalities:

$$F_{A+u^{-1}du}=F_A,$$
$$D_A\left(u^{-1}\psi\right)=u^{-1}D_A\psi. \tag{55}$$

Hence, if the pair (A,ψ) is a solution to the Seiberg-Witten equations, then the pair $(A+u^{-1}du,u^{-1}\psi)$ is also a solution to the Seiberg-Witten equations.

One can obtain infinitely many solutions for the Seiberg-Witten equations on $\mathbb{R}^{4,3}$: Consider the spinor

$$\psi=\left(0,0,\psi_3,i\psi_3,\psi_3,i\psi_3,0,0\right),$$
$$\psi_3\left(x_1,x_2,\ldots,x_7\right)=e^{-(i/2)x_1^2x_2} \tag{56}$$

and the connection 1-form

$$A\left(x_1,x_2,\ldots,x_7\right)=\left(ix_1x_2\right)dx_1+\left(\frac{i}{2}x_1^2\right)dx_2. \tag{57}$$

Since the pair (A,ψ) is a solution on $\mathbb{R}^{4,3}$, the pair $(A+idf,e^{-if}\psi)$ is also a solution, where $u=e^{if}$ and f is a smooth real valued function on $\mathbb{R}^{4,3}$.

The moduli space of Seiberg-Witten equations on the manifold with structure group $G^*_{2(2)}$ is

$$\mathfrak{M}=\frac{\left\{\left(A,\psi\right)\in\mathscr{C}:D_A\psi=0,\ F_A^+=-\left(1/4\right)\sigma\left(\psi\right)^+\right\}}{\mathscr{G}}. \tag{58}$$

Whether the moduli space $\mathfrak{M}$ has similar properties of moduli space of Seiberg-Witten equations on a 4-dimensional manifold is a subject of another work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] E. Witten, "Monopoles and four-manifolds," *Mathematical Research Letters*, vol. 1, no. 6, pp. 769–796, 1994.

[2] J. W. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Princeton University Press, Princeton, NJ, USA, 1996.

[3] N. Değirmenci and N. Özdemir, "Seiberg-Witten-like equations on 7-manifolds with G_2-structure," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 4, pp. 457–461, 2005.

[4] N. Değirmenci and N. Özdemir, "Seiberg-Witten like equations on 8-manifolds with structure group spin(7)," *Journal of Dynamical Systems and Geometric Theories*, vol. 7, no. 1, pp. 21–39, 2009.

[5] Y. H. Gao and G. Tian, "Instantons and the monopole-like equations in eight dimensions," *Journal of High Energy Physics*, vol. 5, article 036, 2000.

[6] T. Nitta and T. Taniguchi, "Quaternionic Seiberg-Witten equation," *International Journal of Mathematics*, vol. 7, no. 5, p. 697, 1996.

[7] N. Değirmenci and N. Özdemir, "Seiberg-Witten like equations on Lorentzian manifolds," *International Journal of Geometric Methods in Modern Physics*, vol. 8, no. 4, 2011.

[8] N. Değirmenci and S. Karapazar, "Seiberg-Witten like equations on Pseudo-Riemannian *Spin^c*-manifolds with neutral signature," *Analele stiintifice ale Universitatii Ovidius Constanta*, vol. 20, no. 1, 2012.

[9] A. Ikemakhen, "Parallel spinors on pseudo-Riemannian Spin^c manifolds," *Journal of Geometry and Physics*, vol. 56, no. 9, pp. 1473–1483, 2006.

[10] H. Baum and I. Kath, "Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds," *Annals of Global Analysis and Geometry*, vol. 17, no. 1, pp. 1–17, 1999.

[11] F. R. Harvey, *Spinors and Calibrations*, Academic Press, 1990.

[12] I. Kath, "$G^*_{2(2)}$-structures on pseudo-riemannian manifolds," *Journal of Geometry and Physics*, vol. 27, no. 3-4, pp. 155–177, 1998.

[13] E. Corrigan, C. Devchand, D. B. Fairlie, and J. Nuyts, "First-order equations for gauge fields in spaces of dimension greater than four," *Nuclear Physics, Section B*, vol. 214, no. 3, pp. 452–464, 1983.

[14] H. B. Lawson and M. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, NJ, USA, 1989.

[15] T. Friedrich, *Dirac Operators in Riemannian Geometry*, American Mathematical Society, 2000.