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On matrix rings with the SIP and the Ads

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Abstract: In this paper, matrix rings with the summand intersection property (SIP) and the absolute direct summand (ads) property (briefly, SA) are studied. A ring R has the right SIP if the intersection of two direct summands of R is also a direct summand. A right R-module M has the ads property if for every decomposition $M = A \oplus B$ of M and every complement C of A in M, we have $M = A \oplus C$. It is shown that the trivial extension of R by M has the SA if and only if R has the SA, M has the ads, and $(1-e)Me = 0$ for each idempotent e in R. It is also shown with an example that the SA is not a Morita invariant property.

Key words: Ads property, summand intersection property, trivial extension

1. Introduction

The purpose of this paper is to study matrix rings that have both the summand intersection property (SIP) and the absolute direct summand (ads) property (briefly, SA).

Wilson [11] defines a right R-module M to have the *SIP* if the intersection of every pair of direct summands of M is a direct summand of M. The ring R has the right SIP provided that the right R-module R has the SIP.

Fuchs [4] introduced the *ads property* for abelian groups. Burgess and Raphael [3] define a right R-module M to have the *ads* if for every decomposition $M = A \oplus B$ of M and every complement C of A in M we have $M = A \oplus C$. The ring R has the right ads provided that the right R-module R has the ads.

Takıl Mutlu [10] defines a right R-module M to have the *SA property* (or briefly have the SA), if M has the SIP and the ads. In [10], the author studied the class of modules with the SA and investigated some properties of these modules. The ring R has the right SA provided that the right R-module R has the SA.

The motivation of the current study comes from the following question:
When does the full matrix ring over a ring have the SA property?

In this paper we provide necessary and sufficient conditions for rings and trivial extensions to have the SA.

Throughout the paper all rings are associative with unity and R always denotes such a ring. Modules are unital and for an abelian group M we use M_R to indicate that M is a right R-module. For any right R-module M, $SocM$ will denote the socle of M. The notions that are not explained here can be found in [12].

We begin with the following lemmas and a proposition that are useful in determining the ads property and the SA property of a module.

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Lemma 1.1 *([3], Proposition 1.1) A module M_R is an ads-module if and only if for any decomposition $M_R = A \oplus B$, B is $A-$injective.*

Proposition 1.2 *([10], Proposition 2.6.) A module M_R has the SA if and only if the following statements are satisfied:*
for any decomposition $M_R = A \oplus B$,

- *i) for every homomorphism f from A to B, the kernel of f is a direct summand.*
- *ii) for any complement C of A in M_R and the projection map $\pi : M \longrightarrow B$, the restricted map $\pi_{|C} : C \longrightarrow B$ is an isomorphism.*

Lemma 1.3 *([10], Lemma 2.7.) Every direct summand of a module that has the SA has again the SA.*

The following example shows that a direct sum of modules that have the SA may not have the SA.

Example 1.4 *This example is taken from [10].*

- *(i) Consider a right $\mathbb{Z}$-module $\mathbb{Z}$. It is clear that $\mathbb{Z}$ is indecomposable and hence it has the SA. Since $\mathbb{Z}$ is not $\mathbb{Z}$-injective, $\mathbb{Z} \oplus \mathbb{Z}$ is not an ads-module by Lemma 1.1 and hence it does not have the SA.*
- *(ii) Consider a right $\mathbb{Z}$-module Prüfer p-group $\mathbb{Z}_{p^\infty}$. It is clear that $\mathbb{Z}_{p^\infty}$ is indecomposable and hence it has the SA. Now define a homomorphism f from $\mathbb{Z}_{p^\infty}$ to $\mathbb{Z}_{p^\infty}$ as the multiplication by p*

$$f\left(\frac{n}{p^t} + \mathbb{Z}\right) = \frac{n}{p^{t-1}} + \mathbb{Z} \text{ with } n \in \mathbb{Z} \text{ and } t \in \mathbb{N}.$$

It is clear that $Kerf = \left(\frac{1}{p} + \mathbb{Z}\right)$. However, $\mathbb{Z}_{p^\infty}$ is indecomposable and hence $Kerf$ is not a direct summand of $\mathbb{Z}_{p^\infty}$. By Proposition 1.2, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ does not have the SA.

Lemma 1.5 *([5], Lemma 3.1) Let R be a product of rings, $R = \prod_I R_i$. Then R has the SIP on the left if and only if each R_i has the SIP on the left.*

2. Rings with the SA property

We shall give the following examples that do not have the SA.

Example 2.1 ***i)*** *Let R be the algebra of matrices, over a field K, of the form*

$$R = \begin{pmatrix} a & x & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & y & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & z \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix}.$$

as in [7]. Let $e = e_{11} + e_{22} + e_{44} + e_{55}$, where e_{ii} denotes the matrix in R with (i,i) entry 1 and all other entries 0. Then e is an idempotent of R and $ReR \neq R$. By [9], $S = eRe$ does not have the ads. Hence, S does not have the SA.

ii) *Let* $R = \left(\begin{smallmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ *be the formal triangular matrix ring. Then the only direct summands (as right ideals) of* R *are* 0*,* R*,* $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$*,* $\left(\begin{smallmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & 0 \end{smallmatrix}\right)$*,* $\left(\begin{smallmatrix} \bar{0} & \bar{1} \\ 0 & 1 \end{smallmatrix}\right)R$*, and* $\left(\begin{smallmatrix} \bar{0} & \bar{2} \\ 0 & 1 \end{smallmatrix}\right)R$*. Since* $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{smallmatrix}\right) \cap \left(\begin{smallmatrix} \bar{0} & \bar{1} \\ 0 & 1 \end{smallmatrix}\right)R$ *is not a direct summand of* R*,* R *does not have the SIP. It follows that* R *does not have the SA.*

Let R be a ring, e an idempotent in R such that $R = ReR$, and S the subring eRe. It is clear that if M is a right R-module, then Me is a right S-module.

Lemma 2.2 *([1], Lemma 5(i)) With the above notation,* C *is a complement of* X *in* M *if and only if* Ce *is a complement of* Xe *in* Me*.*

Proof Suppose that C is a complement of X in M. Since $C \cap X = 0$, $Ce \cap Xe = 0$. Then there exists a complement Ke of Xe in Me such that $Ce \leq Ke$. Hence $C \leq K \leq M$. On the other hand, $(K \cap X)e = 0$. Therefore $(K \cap X)eR = 0$ or $(K \cap X)ReR = (K \cap X)R = 0$. It follows that $K \cap X = 0$. By assumption, $K = C$ and so $Ke = Ce$.

For the converse, assume that Ce is a complement of Xe in Me. We show that C is a complement of X. Since $Ce \cap Xe = 0$, by the above proof, $C \cap X = 0$. Then there exists a complement L of X in M such that $C \leq L$. Then $Ce \leq Le$ and $Le \cap Xe = 0$. As Ce is a complement of Xe in Me, $Le = Ce$. Thus $L = C$. □

Lemma 2.3 *With the above notation,* $M = X \oplus Y$ *if and only if* $Me = Xe \oplus Ye$*.*

Proof Suppose that $M = X \oplus Y$. Since $X \cap Y = 0$, $Xe \cap Ye = 0$. On the other hand, for every $M \in M$, there exist $x \in X$ and $y \in Y$ such that $m = x + y$. Therefore $me = xe + ye$. Hence $Me = Xe \oplus Ye$.

For the converse, assume that $Me = Xe \oplus Ye$. For every $m \in M$, we have $me = xe + ye$ for some $x \in X$ and $y \in Y$. Then $mer_1er_2 = xer_1er_2 + yer_1er_2$ and hence $M_R \leq X_R + Y_R$. It follows that $M_R = X_R + Y_R$. Let $\sum_i x_i r_i = \sum_j y_j s_j$, where $x_i \in X$, $y_j \in Y$, and $r_i, s_j \in R$. For all $r \in R$, we obtain that

$$\sum_i x_i(r_i re) = \sum_j y_j(s_j re) \in Xe \cap Ye.$$

Since $Xe \cap Ye = 0$, we have $\sum_i x_i(r_i re) = 0$, which implies that $(\sum_i x_i r_i)Re = 0$, and by the assumption $ReR = R$, $(\sum_i x_i r_i) = 0$. Thus, $M_R = X_R \oplus Y_R$. □

Theorem 2.4 *With the above notation, if the module* $(Me)_S$ *has the ads, then the module* M_R *has the ads.*

Proof Let $M = X \oplus Y$ and C be a complement of X. Then $Me = Xe \oplus Ye$ and Ce is a complement of Xe in Me by Lemma 2.2 and Lemma 2.3. Since Me has the ads, $Me = Xe \oplus Ce$. By Lemma 2.3, $M = X \oplus C$, i.e. M has the ads. □

Theorem 2.5 *With the above notation, the module* M_R *has the SA if and only if the module* $(Me)_S$ *has the SA.*

Proof By ([6], Theorem 6), the module M_R has the SIP if and only if the module $(Me)_S$ has the SIP. By Theorem 2.4, if the module $(Me)_S$ has the ads, then the module M_R has the ads. To finish the proof, we prove that if the module M_R has the ads, then the module $(Me)_S$ has the ads.

Let $Me = Xe \oplus Ye$ and Ce be a complement of Xe. By Lemma 2.2 and Lemma 2.3, C is a complement of X and $M = X \oplus Y$. Since M has the ads, $M = X \oplus C$. Thus $Me = Xe \oplus Ce$, i.e. Me has the ads. □

Example 2.6 *Let A be a right Ore domain, D the division ring that is the classical right ring of quotients of A, and R the ring of 2×2 matrices over D. Let $e=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in R$. Then e is an idempotent and $ReR=R$. Let $M=R$ and S the subring $eRe=\begin{bmatrix}0 & 0\\ 0 & D\end{bmatrix}$. Hence, $(Me)_S=\begin{bmatrix}0 & D\\ 0 & D\end{bmatrix}$ is indecomposable. Therefore, $(Me)_S$ has the SA and hence M_R has the SA by Theorem 2.5.*

Corollary 2.7 *With the above notation, the ring R has the right SA if and only if the module $(Re)_{eRe}$ has the SA.*

Proof This follows immediately from Theorem 2.5. □

Now let S be a ring, n a positive integer, $M_n(S)$ denote the ring of $n\times n$ matrices over S, and e_{11} be the matrix in $M_n(S)$ with $(1,1)$ entry 1 and all other entries 0. It is well known that e_{11} is idempotent and $S\cong e_{11}M_n(S)e_{11}$ and $M_n(S)=M_n(S)e_{11}M_n(S)$.

Thus, Theorem 2.5 gives the following result, which was mentioned above without proof.

Theorem 2.8 *With the above notation, the ring $M_n(S)$ has the SA if and only if the free module S_S^n has the SA.*

Example 2.9 *Let p be any prime integer and S the polynomial ring $\mathbb{Z}_p[x]$. Consider the ring of 2×2 matrices over $\mathbb{Z}_p[x]$, i.e.*

$$M_2(S)=\begin{bmatrix}\mathbb{Z}_p[x] & \mathbb{Z}_p[x]\\ \mathbb{Z}_p[x] & \mathbb{Z}_p[x]\end{bmatrix}=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\mid a,b,c,d\in\mathbb{Z}_p[x]\right\}.$$

Since $\mathbb{Z}_p$ is a right Noetherian domain, $S=\mathbb{Z}_p[x]$ is also a right Noetherian domain. Hence, $\mathbb{Z}_p[x]$ is a right Ore domain. Thus, by ([2], Proposition 4), $S_S^2=\mathbb{Z}_p[x]\oplus\mathbb{Z}_p[x]$ has the SIP. On the other hand, since $f(x)\in\mathbb{Z}_p[x]$ is idempotent iff its constant term is idempotent and other coefficients are zero, $\mathbb{Z}_p[x]$ is indecomposable. Then $S_S^2=\mathbb{Z}_p[x]\oplus\mathbb{Z}_p[x]$ is indecomposable and hence it has the ads. Finally, S_S^2 has the SA. Then, by Theorem 2.8, $M_2(S)$ has the SA.

Recall that a ring theoretic property $\mathcal{P}$ is said to be a Morita invariant property if and only if all the following hold:
whenever a ring R has $\mathcal{P}$ then

i. $M_n(R)$ has $\mathcal{P}$ for all $n\geq 2$,

ii. eRe has $\mathcal{P}$ for all $e^2=e\in R$ such that $R=ReR$.

The following example shows that the SA property is not a Morita invariant property.

Example 2.10 *Let $R=\mathbb{Z}$. Consider the ring $R^2=\mathbb{Z}\times\mathbb{Z}$. Since $\mathbb{Z}\times\{0\}$ is not $(\{0\}\times\mathbb{Z})$-injective, R^2 does not have the ads by Lemma 1.1. Hence, $M_2(R)$ does not have the SA by Theorem 2.8. Thus, the SA property is not a Morita invariant property.*

Given a ring R and a $R-R$-bimodule M, the trivial extension of a ring R by M is defined to be the ring whose additive group is the direct sum $R \oplus M$ with multiplication given by

$$(r, m) \cdot (r^{'}, m^{'}) = (rr^{'}, rm^{'} + mr^{'}).$$

Now we give the following result on trivial extensions.

Theorem 2.11 *Let R be any ring, M be an $R-R$-bimodule, and T be the corresponding trivial extension of R by M. Then T has the SA if and only if all the following hold:*

(i) R has the SA,

(ii) M has the ads,

(iii) $(1-e)Me = 0$ for each idempotent e of R.

Proof Suppose that T has the SA. Then, by ([6], Theorem 11) and ([9], Theorem 3.6), (i), (ii), and (iii) hold.

Assume that (i), (ii), and (iii) hold. Then, by ([6], Theorem 11), T has the SIP and each direct summand of T is (eR, eM) for some idempotent element e of R. To finish the proof, we show that T has the ads. Let $T = (eR, eM) \oplus (fR, fM)$ for some idempotent element e, f of R and (K, L) a complement of (eR, eM). Then K is a complement of eR and L is a complement of eM. By hypothesis, $R = eR \oplus K$ and $M = eM \oplus L$, and so $T = (eR, eM) \oplus (K, L)$. Hence, T has the ads, as desired. □

Example 2.12 *Let $R = \mathbb{Z}_6$. Consider the R-R bimodule $M = 2\mathbb{Z}_6$. Let*

$$T = \begin{bmatrix} \mathbb{Z}_6 & 2\mathbb{Z}_6 \\ 0 & \mathbb{Z}_6 \end{bmatrix} = \left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \; r \in \mathbb{Z}_6, \; m \in 2\mathbb{Z}_6 \right\}.$$

denote the trivial extension of R by M. Since $\mathbb{Z}_6$ is semisimple, the only nontrivial decomposition of R_R is $2\mathbb{Z}_6 \oplus 3\mathbb{Z}_6$ and $2\mathbb{Z}_6$ is $3\mathbb{Z}_6$-injective, R_R has the SA by Lemma 1.1, and hence $\mathbb{Z}_6$ and $2\mathbb{Z}_6$ have the SA by Lemma 1.3. On the other hand, all idempotent elements of $\mathbb{Z}_6$ are $\overline{0}$, $\overline{1}$, $\overline{3}$, and $\overline{4}$ and for each idempotent e of $\mathbb{Z}_6$, $(1-e)Me = 0$. Then, by Theorem 2.11, T has the SA.

From now on this paper, let T be the formal triangular matrix ring $\left[\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right]$, where R and S are rings with identities and M is a $S-R$-bimodule.

Theorem 2.13 *If T has the SA, then R has the SA.*

Proof The result is a consequence of ([6], Theorem 15) and ([9], Theorem 3.6). □

Note that the converse of Theorem 2.13 is not always true. In fact, let $T = \left[\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z}\oplus\mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$. The $\mathbb{Z} \oplus \mathbb{Z}$ as a right $\mathbb{Z}$-module is not ads since $\mathbb{Z}$ is not $\mathbb{Z}$-injective. Thus, T does not have the SA by ([9], Theorem 3.6), although $\mathbb{Z}$ has the ads.

Theorem 2.14 *Let $SocT$ be a direct summand of T. If T has the SA, then both R and S have the SA.*

Proof Since $SocT$ is a direct summand of T by assumption, $M = 0$ by ([6], Lemma 12). Hence, $T \cong S \times R$. Then R and S have the SA by Lemma 1.3. □

Now we provide the following example for comparison to Theorem 2.14.

Example 2.15 *Let F be a field and T the formal triangular matrix ring over F, i.e.*

$$\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in F \right\}.$$

Routine calculations show that any nontrivial idempotent of T has one of the following forms, where $f \in F$:

$$c = \begin{bmatrix} 0 & f \\ 0 & 1 \end{bmatrix}, e = \begin{bmatrix} 1 & f \\ 0 & 0 \end{bmatrix}.$$

It can be easily seen that $cT \cap eT = 0$. Hence, T has the SIP. Consider the decomposition $T = cT \oplus eT$. One can then verify that the only complement of cT is $A = \left[\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right]$ and the complement of eT has one of the following forms:

$$B = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix},\ C = \left\{ \begin{bmatrix} 0 & fx \\ 0 & x \end{bmatrix} \mid 0 \neq x, f \in F \right\}.$$

Moreover, $T = cT \oplus A$, $T = eT \oplus B$, and $T = eT \oplus C$. Hence, T has the ads. Therefore, T has the SA. On the other hand, F has the SA and $SocT = \left[\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right]$ is not a direct summand of T.

Theorem 2.16 *Let R and S be mutually injective CS rings. Assume that $SocT$ is a direct summand of T. Then T has the SA if and only if R and S have the SA.*

Proof ($\Longrightarrow$) It is clear from Theorem 2.14.
($\Longleftarrow$) Let R and S have the SA. Then $S \times R$ has the SIP by Lemma 1.5. Furthermore, since both R and S are CS and ads, both R and S are quasicontinuous. Hence, $S \times R$ is quasicontinuous by ([8], Corollary 2.14). Therefore, $S \times R$ has the ads by ([8], Theorem 2.8). It follows that T has the SA. □

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