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Derivatives of the restrictions of harmonic functions on the Sierpinski gasket to segments

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Abstract

We give an explicit derivative computation for the restriction of a harmonic function on SG to segments at specific points of the segments: The derivative is zero at points dividing the segment in ratio 1:3. This shows that the restriction of a harmonic function to a segment of SG has the following curious property: The restriction has infinite derivatives on a dense subset of the segment (at junction points) and vanishing derivatives on another dense subset.
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We will first briefly recall the rudiments of harmonic analysis on the Sierpinski gasket [1–4].

Let K be the Sierpinski gasket (SG) constructed on the unit equilateral triangle G_0 with vertices $\{p_0, p_1, p_2\}$ and G_m be the graph in the mth step as in Fig. 1.

Definition 1. The function $f \in C(K)$, $f : K \to \mathbb{R}$ is called harmonic on K if for every minimal triangle in G_m $(m \geqslant 1)$, with vertices $\{v_i, v_j, v_k\}$, the equalities

$$f(v_i) + f(v_j) + f(v_{ik}) + f(v_{jk}) - 4f(v_{ij}) = 0 \tag{1}$$

hold, where v_{ij} is the midpoint of the segment $[v_i, v_j]$.

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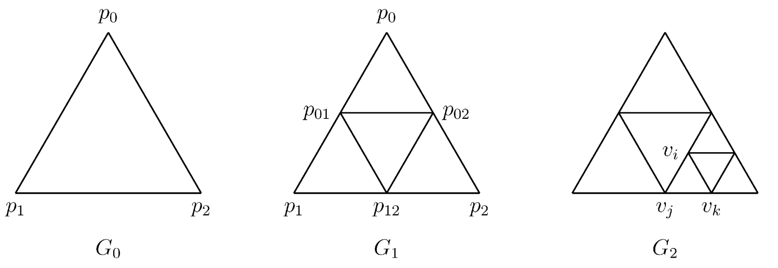

Fig. 1. Iterated graphs in SG.

Let

$$f(p_0) = \alpha, \qquad f(p_1) = \beta, \qquad f(p_2) = \gamma. \tag{2}$$

Then this triple (α, β, γ) completely defines a harmonic function f, that is, there exists a unique harmonic function $f : K \to \mathbb{R}$ such that $f(p_0) = \alpha$, $f(p_1) = \beta$ and $f(p_2) = \gamma$. This harmonic function depends linearly on the triple (α, β, γ). According to the harmonic extension algorithm (which can be obtained from (1)), it holds

$$f(p_{12}) = \frac{1}{5}(\alpha + 2\beta + 2\gamma), \qquad f(p_{02}) = \frac{1}{5}(2\alpha + \beta + 2\gamma),$$
$$f(p_{01}) = \frac{1}{5}(2\alpha + 2\beta + \gamma). \tag{3}$$

From (1)–(3) it can be seen that, if a nonconstant harmonic function is monotone on some line segment that is contained in SG, then it is strictly monotone on it. (In the following we discard constant functions.)

Let T_m be a minimal triangle with vertices v_i, v_j and v_k in G_m. The sides of T_m can be ordered by the values $|f(v_i) - f(v_j)|$.

Theorem 2. [2] *The restriction of f to the two largest edges of T_m is monotone. On the smallest edge of T_m, the restriction of f might be monotone or not*; *but if it is not monotone, then it has a unique extremum.* (*We changed the wording of the Theorem* 2 *in* [2] *slightly.*)

It is more or less folklore, that the derivatives at the junction points of any segment E in G_m (of the restriction of a nonconstant harmonic function to that segment) exist improperly (possibly with exception of a single point), and we will give for convenience a proof of this fact. But our main goal will be to show that there exists another dense subset of E, on which the derivatives of the restriction vanish.

We remark that it is enough to prove these statements for the triangle G_0 instead of considering an arbitrary triangle T_m in G_m, because the procedure of harmonic extension is the same for G_0 or T_m.

Now, consider the side $[p_1, p_2] = [0, 1]$ of G_0 and the restriction of the harmonic function f defined by (2) to $[p_1, p_2]$. The following lemma can be proved by induction on m.

Lemma 3. *Let* $l_m = \frac{1}{2} - \frac{1}{2^{m+1}}$, $r_m = \frac{1}{2} + \frac{1}{2^{m+1}}$ $(m = 1, 2, 3, \ldots)$. *Then*

$$f\left(\frac{1}{2^m}\right) = \frac{3^m - 1}{2 \cdot 5^m}\alpha + \left[1 - \left(\frac{3}{5}\right)^m\right]\beta + \frac{3^m + 1}{2 \cdot 5^m}\gamma, \tag{4}$$

$$f\left(1-\frac{1}{2^m}\right)=\frac{3^m-1}{2\cdot 5^m}\alpha+\left[1-\left(\frac{3}{5}\right)^m\right]\gamma+\frac{3^m+1}{2\cdot 5^m}\beta, \tag{5}$$

$$f(l_m)=\frac{5^m-1}{5^{m+1}}\alpha+\frac{3^{m+1}+4\cdot 5^m+3}{10\cdot 5^m}\beta+\frac{4\cdot 5^m-3^{m+1}-1}{10\cdot 5^m}\gamma, \tag{6}$$

$$f(r_m)=\frac{5^m-1}{5^{m+1}}\alpha+\frac{3^{m+1}+4\cdot 5^m+3}{10\cdot 5^m}\gamma+\frac{4\cdot 5^m-3\cdot 3^{m+1}-1}{10\cdot 5^m}\beta. \tag{7}$$

(Actually, by symmetry, one of these equalities implies the other three.)

We first consider the junction points and need the following

Lemma 4. *Let the function $g:[0,1]\to\mathbb{R}$ be strictly monotone in a neighborhood of $x_0\in[0,1]$, $d\in(0,1)$, $a\neq 0$ and $x_m=x_0+ad^m$. Assume*

$$\frac{g(x_m)-g(x_0)}{x_m-x_0}$$

is defined and tends to 0 (or $\pm\infty$) as $m\to\infty$. If $a<0$ then the left derivative of g at x_0 exists and is 0 (or $\pm\infty$); if $a>0$ then the right derivative of g at x_0 exists and is 0 (or $\pm\infty$).

Proof. We consider only the case, where g is monotone increasing and $a>0$. Let $x_0\in[0,1)$ and $x>x_0$. Then there exists $m\in\mathbb{N}$ such that

$$x_0+ad^{m+1}\leqslant x\leqslant x_0+ad^m.$$

As x tends to x_0, m tends to infinity and from the inequalities

$$d\cdot\frac{g(x_{m+1})-g(x_0)}{x_{m+1}-x_0}\leqslant\frac{g(x)-g(x_0)}{x-x_0}\leqslant\frac{1}{d}\cdot\frac{g(x_m)-g(x_0)}{x_m-x_0}$$

we get the result. □

Remark 5. In the above lemma, one-sided monotonicity is obviously enough for one-sided derivative calculations.

We can now compute the derivative of the restriction at the point $p=1/2$, for monotone restrictions.

Lemma 6. *Let the restriction of the harmonic function f to the edge $[p_1,p_2]=[0,1]$ be strictly monotone. Then $f'(\frac{1}{2})=+\infty$ for f monotone increasing and $f'(\frac{1}{2})=-\infty$ for f monotone decreasing.*

Proof. We give the proof for f monotone increasing.

Using (6), we obtain

$$\lim_{m\to\infty}\frac{f(l_m)-f(\frac{1}{2})}{l_m-\frac{1}{2}}=\lim_{m\to\infty}\left(\frac{3}{5}\right)\cdot\left(\frac{6}{5}\right)^m(\gamma-\beta)=+\infty.$$

Then by Lemma 4 (with $x_0=\frac{1}{2}, a=-\frac{1}{2}, d=\frac{1}{2}$) the left-hand derivative at $p=\frac{1}{2}$ is $+\infty$. Analogously, using (7), we obtain that the right-hand derivative at $p=\frac{1}{2}$ is also $+\infty$. □

Applying Lemma 6 to smaller triangles, we see that the derivatives exist improperly at all inner junction points of $[p_1, p_2]$ in whose vicinity the restriction is strictly monotone.

We now come to our main point and we will show that the derivative of the restriction of a harmonic function f on SG to an edge of any G_m is differentiable at a point dividing the edge in ratio 1:3 and that the derivative there vanishes. It is again enough to show this for the edge $[p_1, p_2] = [0, 1]$ of G_0 as the extension rule for the harmonic function is the same at every scale.

Theorem 7. *Let f be a harmonic function on SG and p the point dividing the edge $[p_1, p_2]$ in ratio* 1:3 (*i.e.* $p = 1/3$). *Then*

$$(f|_{[0,1]})'\left(\frac{1}{3}\right) = 0.$$

Proof. Let us first assume that the restriction of f to $[0, 1]$ is monotone increasing. To approach the point $p = \frac{1}{3}$ from left and right with geometrically convergent sequences we use the following sequence of triangles $\triangle_m = \{p_0^m, p_1^m, p_2^m\}$.

Let $\triangle_0 = G_0 = \{p_0, p_1, p_2\}$ and let $\triangle_m$ be defined as in Fig. 2 (right third of the left third of $\triangle_{m-1}$).

One can compute

$$p_1^m = \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^m = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{4}\right)^m,$$
$$p_2^m = p_1^m + \left(\frac{1}{4}\right)^m = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{4}\right)^m.$$

Let $f(p_0^m) = \alpha_m$, $f(p_1^m) = \beta_m$ and $f(p_2^m) = \gamma_m$, $\alpha_0, \beta_0, \gamma_0$ being α, β, γ.

We want to compute the values β_m and γ_m explicitly. Using (3) we get

$$\alpha_m = \frac{1}{25}[6\alpha_{m-1} + 13\beta_{m-1} + 6\gamma_{m-1}], \tag{8}$$
$$\beta_m = \frac{1}{25}[4\alpha_{m-1} + 16\beta_{m-1} + 5\gamma_{m-1}], \tag{9}$$
$$\gamma_m = \frac{1}{5}[\alpha_{m-1} + 2\beta_{m-1} + 2\gamma_{m-1}]. \tag{10}$$

From (8)–(10) we obtain

$$5\alpha_m + 15\beta_m + 7\gamma_m = 5\alpha_{m-1} + 15\beta_{m-1} + 7\gamma_{m-1}$$

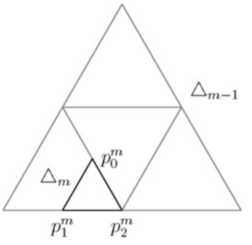

Fig. 2. The sequence of triangles $\triangle_m$.

for all $m = 1, 2, \ldots$. In other words,

$$5\alpha_m + 15\beta_m + 7\gamma_m = 5\alpha + 15\beta + 7\gamma =: c. \tag{11}$$

From (11) and continuity of f we get

$$f\left(\frac{1}{3}\right) = \frac{c}{27}. \tag{12}$$

Using (11) we can eliminate α_{m-1} from (10):

$$\beta_m = \frac{1}{125}[4c + 20\beta_{m-1} - 3\gamma_{m-1}], \tag{13}$$

$$\gamma_m = \frac{1}{125}[5c - 25\beta_{m-1} + 15\gamma_{m-1}]. \tag{14}$$

As can be seen from (13) and (14), the sequence

$$t_m = u\beta_m + v\gamma_m \tag{15}$$

with $u = 10$, $v = 1 - \sqrt{13}$, satisfies the recursion formula

$$t_m = w + st_{m-1}, \tag{16}$$

where $w = \frac{9-\sqrt{13}}{25}c$, $s = \frac{7+\sqrt{13}}{50}$.

From (16) t_m can be determined:

$$t_m = w\frac{s^m - 1}{s - 1} + s^m \cdot t_0 \quad \bigl(t_0 = 10\beta + (1 - \sqrt{13})\gamma\bigr). \tag{17}$$

From (13), (14) and (16) we obtain

$$\gamma_m = \frac{c}{25} - \frac{1}{50}t_{m-1} + \frac{v+6}{50}\gamma_{m-1},$$

and inserting t_{m-1} from (17) we get

$$\gamma_m = l + k \cdot s^{m-1} + h \cdot \gamma_{m-1}, \tag{18}$$

where $l = \frac{c}{25} + \frac{w}{50(s-1)}$, $k = -\frac{1}{50}(\frac{w}{s-1} + t_0)$, $h = \frac{v+6}{50}$.

The recursion (18) gives γ_m explicitly:

$$\gamma_m = \left[\frac{l}{h-1} - \frac{k}{s-h} + \gamma\right]h^m + \frac{k}{s-h}s^m + \frac{c}{27}.$$

As $0 < h < \frac{1}{4}$ and $0 < s < \frac{1}{4}$ we obtain finally

$$\lim_{m\to\infty} \frac{f(p_2^m) - f(\frac{1}{3})}{p_2^m - \frac{1}{3}} = 0.$$

Taking $x_0 = \frac{1}{3}$, $d = \frac{1}{4}$ and $a = \frac{2}{3}$ in Lemma 4, we see that the right derivative of the restriction of f to $[p_1, p_2] = [0, 1]$ at $p = 1/3$ exists and is zero.

Similarly, from (13), (14), (16) we get

$$\beta_m = \frac{1}{u}\left[\frac{w}{s-1} - \frac{vk}{s-h} + t_0\right] \cdot s^m - \frac{v}{u}\left(\frac{l}{h-1} - \frac{k}{s-h} + \gamma\right) \cdot h^m + \frac{c}{27}$$

and this shows that the left derivative at $p = \frac{1}{3}$ exists and is also zero. Together we obtain

$$(f|_{[0,1]})'\left(\frac{1}{3}\right) = 0.$$

Now we consider the case where the restriction of f to [0, 1] is not monotone. In that case we know that the restriction is monotone in two pieces. If the extremum is not attained at $p = 1/3$, then there is a neighborhood $(\frac{1}{3} - \delta, \frac{1}{3} + \delta)$ where the restriction is monotone and the above proof applies. If the extremum is attained at $p = 1/3$, then Lemmas 3, 4 and the above proof works still on two sides of $p = 1/3$ and we get $(f|_{[0,1]})'(\frac{1}{3}) = 0$. □

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