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A NOTE ON SEIBERG-WITTEN MONOPOLE EQUATIONS ON $\mathbf{R}^8$

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ABSTRACT

Salamon's generalizations of the Seiberg-Witten equations are meaningful on any even-dimensional manifolds. In this work we show that there are no nontrivial solutions of these equations for any spin^c-structures on $\mathbf{R}^8$.

1. INTRODUCTION

The Seiberg-Witten monopole equations are stated for 4-dimensional manifolds and these equations have great importance for the topology of smooth four-manifolds (see [7], [5]). There are also some analogous to these equations in 8-dimension (see [2], [7], [3]). In [1] it is shown that the one given by Salamon [7] have no nontrivial solutions for the standart spin^c-structures on $\mathbf{R}^8$. In this work we show that Salamon's generalization of the Seiberg-Witten equations have no nontrivial solutions for any spin^c-structures on $\mathbf{R}^8$.

2. PRELIMINARIES

In this section we give some basic definitions and facts about Seiberg-Witten monopole equations. For more details one can look in [7].

Definition 2.1. A spin^c-structure on a $2n$-dimensional oriented real Hilbert space V is a pair (W,Γ) where W is a 2^n-dimensional complex Hermitian vector space and $\Gamma : V \rightarrow End(W)$ is a linear map which satisfies

$$\Gamma(v)^* + \Gamma(v) = 0, \qquad \Gamma(v)^*\Gamma(v) = \|v\|^2$$

for every $v \in V$.

It is pointed out in [7] that such a map can be extended to an algebra isomorphism $Cl(V) \rightarrow End(W)$ which satisfies $\Gamma(\widetilde{x}) = \Gamma(x)^*$, where

$\mathbf{C}l(V) \cong Cl(V) \otimes \mathbf{C}$ is complex Clifford algebra over V, $\widetilde{x}$ is conjugate of x in $\mathbf{C}l(V)$ and $\Gamma(x)^*$ denotes hermitian-conjugate of $\Gamma(x)$.

Let (W_1, Γ_1) and (W_2, Γ_2) be two spin^c-structures on V. If there exists a unitary isomorphism $U : W_1 \to W_2$ such that

$$U\Gamma_1(v)U^* = \Gamma_2(v)$$

for all $v \in V$, then the spin^c-structures (W_1, Γ_1) and (W_2, Γ_2) are said to be equivalent. It is known that such a unitary isomorphism always exists as a result of the following proposition (see [7]).

Proposition 2.2. Let (W_1, Γ_1) and (W_2, Γ_2) be two spin^c-structures on V. Then there exists a unitary isomorphism $U : W_1 \to W_2$ such that

$$U\Gamma_1(v)U^* = \Gamma_2(v)$$

for all $v \in V$.

Let (W, Γ) be a spin^c-structure on V. There is a natural splitting of W. Fix an orientation of V and denote by

$$\varepsilon = e_{2n} \cdots e_2 e_1 \in Cl(V)$$

the unique element of $Cl(V)$ which has degree $2n$ and is generated by a positively oriented orthonormal basis $e_1, \cdots, e_{2n}$. Then $\varepsilon^2 = (-1)^n$ and hence

$$W = W^+ \oplus W^-$$

where the $W^{\pm}$ are the eigen spaces of $\Gamma(\varepsilon)$

$$W^{\pm} = \{w \in W : \Gamma(\varepsilon)w = \pm i^n w\}.$$

Note that $\Gamma(v)W^+ \subset W^-$ and $\Gamma(v)W^- \subset W^+$ for every $v \in V$. So the restriction of $\Gamma(v)$ to W^+ for $v \in V$ determines a linear map $\gamma : V \to Hom(W^-, W^+)$ which satisfies

$$\gamma(v)^* \gamma(v) = |v|^2 1$$

for every $v \in V$.

Let (W, Γ) be a spin^c structure on V. Such a structure gives an action of the space of 2-forms $\Lambda^2 V$ on W. This action is defined by the following:

Firstly, identify $\Lambda^2 V$ with the space of second order elements of Clifford algebra $C_2(V)$ via the map

$$\Lambda^2 V \to C_2(V), \eta = \sum_{i<j} \eta_{ij} e_i \wedge e_j \mapsto \sum_{i<j} \eta_{ij} e_i e_j .$$

Compose this map with Γ to obtain a map $\rho : \Lambda^2 V \to End(W)$ given by

$$\rho\left(\sum_{i<j} \eta_{ij} e_i \wedge e_j \right) = \sum_{i<j} \eta_{ij} \Gamma(e_i) \Gamma(e_j)$$

for any orthonormal basis $e_1, \cdots, e_{2n}$ of V. This map is independent of the choice of the orthonormal basis $e_1, \cdots, e_{2n}$. The spaces $W^{\pm}$ are invariant under $\rho(\eta)$ for every 2-form $\eta \in \Lambda^2 V$. So we can define

$$\rho^{\pm}(\eta) = \rho(\eta)|_{W^{\pm}}$$

for $\eta \in \Lambda^2 V$. In 4-dimensions $\rho^{+}(\eta) = \rho^{+}(\eta^{+})$ for every 2-form $\eta \in \Lambda^2 V$, where η^{+} is the self-dual part of η. The map ρ extends to a map

$$\rho : \Lambda^2 V \otimes \mathbf{C} \to End(W)$$

on the space of complex valued 2-forms. If η is a real valued 2-form, then $\rho(\eta)$ is skew-Hermitian and if η is imaginary valued then $\rho(\eta)$ is Hermitian.

Globalizing above Γ to 2n-dimensional oriented manifold X defines a spin^c structure $\Gamma : TX \to End(W)$, W being a 2^n-dimensional complex Hermitian vector bundle on X. Such a structure exists iff $w_2(X)$ has an integral lift (see [4]). Γ extends to an isomorphism between the complex Clifford algebra bundle $\mathbf{C}l(TX)$ and $End(W)$. There is a natural splitting $W = W^{+} \oplus W^{-}$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n} e_{2n-1} \cdots e_1)$ where $e_1, e_2, \cdots e_{2n}$ is any positively oriented local orthonormal frame of TX.

A Hermitian connection ∇ on W is called a spin^c connection (compatible with the Levi-Civita connection) if

$$\nabla_v (\Gamma(w)\Psi) = \Gamma(w) \nabla_v \Psi + \Gamma(\nabla_v w)\Psi$$

where Ψ is a spinor (section of W), v and w are vector fields on X and $\nabla_v w$ is the Levi-Civita connection on X. ∇ preserves the subbundles $W^{\pm}$.

There is a principal Spin$^c(2n)$-bundle P on X such that the bundle W of spinors, the tangent bundle TX, and the line bundle L_Γ can be recovered as the associated bundles

$$W = P \times_{Spin^c(2n)} \mathbf{C}^{2n}, \qquad TX = P \times_{Ad} \mathbf{R}^{2n}$$

where Ad is the adjoint action of

$$Spin^c(2n) = \{e^{i\theta} x : \theta \in \mathbf{R}, x \in Spin(2n)\} \subset \mathbf{Cl}_{2n}$$

on R^{2n}. Then one can obtain a complex line bundle $L_\Gamma = P \times_\delta \mathbf{C}$ where

$$\delta : Spin^c(2n) \to S^1 \text{ by } \delta(e^{i\theta} x) = e^{2i\theta}.$$

There is a one-to-one correspondence between spin^c connections on W and $spin^c(2n) = Lie(Spin^c(2n)) = spin(2n) \oplus i\mathbf{R}$-valued connection 1-forms $\widehat{A} \in \mathbf{A}(P) \subset \Omega^1(P, spin^c(2n))$ on P. Hence every spin^c connection $\widehat{A}$ decomposes as

$$\widehat{A} = \widehat{A}_0 + \frac{1}{2^n} trace(\widehat{A})$$

where $\widehat{A}_0$ is the traceless part of $\widehat{A}$. Let $A = \frac{1}{2^n} trace(\widehat{A})$. This is an imaginary valued 1-form in $\Omega^1(P, i\mathbf{R})$ which satisfies

$$A_{pq}(vg) = A_p(v), \qquad A_p(p.\xi) = \frac{1}{2^n} trace(\xi)$$

(1)

for $v \in T_p P$, $g \in Spin^c(2n)$, and $\xi \in spin^c(2n)$. Let

$$\mathbf{A}(\Gamma) = \{A \in \Omega^1(P, i\mathbf{R}) : A \text{ satisfies } (1)\}$$

There is a one-to-one correspondence between these 1-forms and spin^c connections on W. Let ∇_A be the spin^c connection corresponding to A. $\mathbf{A}(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X, i\mathbf{R})$. Let $F_A \in \Omega^2(P, i\mathbf{R})$ be the curvature of the 1-form A and D_A denote the Dirac operator corresponding to $A \in \mathbf{A}(\Gamma)$,

$$D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$$

defined by

$$D_A(\Psi)=\sum_{i=1}^{2n}\Gamma(e_i)\nabla_{A,e_i}(\Psi)$$

where $\Psi \in C^\infty(X, W^+)$ and $e_1, e_2, \cdots, e_{2n}$ is any local orthonormal frame.
The Seiberg-Witten equations can now be expressed as follows:
Let $\Gamma : TX \to End(W)$ be a fixed spin^c structure on X and consider the pair $(A,\Psi) \in \mathbf{A}(\Gamma)\times C^\infty(X,W^+)$. The Seiberg-Witten equations read

$$D_A\Psi = 0, \qquad \rho^+(F_A) = (\Psi\Psi^*)_0$$

where $(\Psi\Psi^*)_0 \in C^\infty(X, End(W^+))$ is defined by $(\Psi\Psi^*)(\tau) = \langle \Psi, \tau\rangle\Psi$ for $\tau \in C^\infty(X, W^+)$ and $(\Psi\Psi^*)_0$ is the traceless part of $(\Psi\Psi^*)$.

3. MONOPOLE EQUATIONS ON R^8 WITH DIFFERENT Spin^c-STRUCTURES AND THEIR RELATIONS

One can find the explicit expressions of the Seiberg-Witten monopole equations on $\mathbf{R}^4$ in [6] and [7].

In our case $X = \mathbf{R}^8$, $W_1 = W_2 = \mathbf{C}^{16}$ and $L_\Gamma = \mathbf{R}^8 \times \mathbf{C}$, (W_1, Γ_1) and (W_2, Γ_2) spin^c-structures on $\mathbf{R}^8$ and we consider the unitary map U from W_1 to W_2 that satisfies

$$U \circ \Gamma_1(v) \circ U^* = \Gamma_2(v) \tag{2}$$

for all $v \in \mathbf{R}^8$.

In [1] they consider standard spin^c-structure which is obtained from the well-known isomorphism of the complex Clifford algebra $\mathbf{C}l_{2n}$ with $End(\Lambda^*\mathbf{C}^n)$ and they express following theorem:

Theorem 3.1. There are no nontrivial solutions of the Seiberg-Witten equations on $\mathbf{R}^8$ with constant standard spin^c-structure, i. e. $\rho^+(F_A) = (\Psi\Psi^*)_0$ (alone) implies $F_A = 0$ and $\Psi = 0$.

Our goal is to state a similar theorem for any spin^c-structure on $\mathbf{R}^8$. To do this we need some lemmas.

Lemma 3.2. If a unitary isomorphism U from W_1 to W_2 satisfies (2), then U maps $W_1^{\pm}$ onto $W_2^{\pm}$.

Proof. Let $\Psi \in C^\infty(\mathbf{R}^8, W_1^+)$. Then $\Gamma_1(\varepsilon)\Psi = \Psi$ where $\varepsilon = e_{2n} \cdots e_1$.

$$\begin{aligned}\Psi &= \Gamma_1(e_{2n}\cdots e_1)\Psi \\ &= \Gamma_1(e_{2n})\cdots\Gamma_1(e_1)\Psi \\ &= U^*\Gamma_2(e_{2n})U\cdots U^*\Gamma_2(e_1)U\Psi \\ &= U^*\Gamma_2(e_{2n}\cdots e_1)U\Psi\end{aligned}$$

From the last equality $\Gamma_2(e_{2n}\cdots e_1)U\Psi = U\Psi$ that is, $U\Psi \in C^\infty(\mathbf{R}^8, W_2^+)$. Thus U maps W_1^+ onto W_2^+. It can be shown in a similar way that U maps W_1^- onto W_2^-.

Lemma 3.3. The maps $\rho_1 : \Lambda^2(T^*\mathbf{R}^8)\otimes\mathbf{C} \to End(W_1)$ and $\rho_2 : \Lambda^2(T^*\mathbf{R}^8)\otimes\mathbf{C} \to End(W_2)$ satisfy $\rho_1(\eta) = U\rho_2(\eta)U^*$ for any 2-form $\eta = \sum_{i<j}\eta_{ij}e_i \wedge e_j$ in $\Lambda^2(T^*\mathbf{R}^8)\otimes\mathbf{C}$.

Proof.

$$\begin{aligned}\rho_2(\eta) &= \sum_{i<j}\eta_{ij}\Gamma_2(e_i)\Gamma_2(e_j) \\ &= \sum_{i<j}\eta_{ij}U\Gamma_2(e_i)U^*U\Gamma_2(e_j)U^* \; (Since\, UU^* = I) \\ &= \sum_{i<j}U\eta_{ij}\Gamma_2(e_i)\Gamma_2(e_j)U^* \\ &= U\left(\sum_{i<j}\eta_{ij}\Gamma_2(e_i)\Gamma_2(e_j)\right)U^* \\ &= U(\rho_1(\eta))U^*.\end{aligned}$$

Note that $\rho_2^+(\eta) = U(\rho_1^+(\eta))U^*$.

Lemma 3.4. If $\Psi \in C^\infty(\mathbf{R}^8, W_1^+)$, then the equality $((U\Psi)(U\Psi)^*)_0 = U(\Psi\Psi^*)_0 U^*$ holds for any unitary isomorphism $U : \mathbf{C}^{16} \to \mathbf{C}^{16}$.

Proof.

$$\begin{aligned}
(U(\Psi\Psi^*)_0 U^*)\tau &= (U(\Psi\Psi^*)_0)(U^*\tau) \\
&= U\langle \Psi, U^*\tau\rangle\Psi - trace(\Psi\Psi^*)U^*\tau \\
&= \langle \Psi, U^*\tau\rangle U\Psi - trace(\Psi\Psi^*)\tau \\
&= \langle \Psi, U^*\tau\rangle U\Psi - trace((U\Psi)(U\Psi)^*) \\
&= ((U\Psi)(U\Psi)^*)_0 \tau
\end{aligned}$$

for all $\tau \in C^\infty(\mathbf{R}^8, W_1^+)$. Note that, $trace(\Psi\Psi^*) = \|\Psi\|^2 = \|U\Psi\|^2 = trace((U\Psi)(U\Psi)^*)$, since U is unitary.

Lemma 3.5. Let (Γ_1, W_1) and (Γ_2, W_2) be two spin^c-structures on $\mathbf{R}^8$ and $U : W_1 \to W_2$ be a unitary isomorphism such that $U \circ \Gamma_1(\nu) \circ U^* = \Gamma_2(\nu)$ for all $\nu \in \mathbf{R}^8$. If the pair (A, Ψ) is a solution of the monopole equations with respect to Γ_1, then the pair $(A, U\Psi)$ is a solution of the monopole equations with respect to Γ_2.

Proof. Let (A, Ψ) be a solution of the equations

$$\begin{aligned}
D_A\Psi &= \sum_{i=1}^{8} \Gamma_1(e_i)\nabla_i(\Psi) = 0 . \\
\rho_1^+(F_A) &= \sum_{i<j} F_{ij}\Gamma_1(e_i)\Gamma_1(e_j) = (\Psi\Psi^*)_0
\end{aligned}$$

Then

$$\begin{aligned} D_A(U\Psi) &= \sum_{i=1}^{8}\Gamma_2(e_i)\nabla_i(U\Psi) \\ &= \sum_{i=1}^{8}U\Gamma_1(e_i)U^*\nabla_i(U\Psi) \\ &= \sum_{i=1}^{8}U\Gamma_1(e_i)U^*U\nabla_i(\Psi) \quad (\text{since}\,\nabla_i(U\Psi)=U\nabla_i(\Psi)) \\ &= U\sum_{i=1}^{8}\Gamma_1(e_i)\nabla_i(\Psi)=U(D_A\Psi)=0. \end{aligned}$$

The equality $\nabla_i(U\Psi)=U\nabla_i(\Psi)$ holds for all $\Psi\in C^\infty(\mathbf{R}^8,W_1^+)$, $U\Psi=\left(\sum_{i=1}^{16}u_{1i}\psi_i,\cdots,\sum_{i=1}^{16}u_{(16)i}\psi_i\right)$ where $U=(u_{ij})$ is the matrix notation of the unitary map U.

$$\nabla_i(U\Psi) = \nabla_i \begin{bmatrix} u_{11}\psi_1 + \cdots + u_{1(16)}\psi_{(16)} \\ u_{21}\psi_1 + \cdots + u_{2(16)}\psi_{(16)} \\ \vdots \\ u_{(16)1}\psi_1 + \cdots + u_{(16)(16)}\psi_{(16)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_i}\left(u_{11}\psi_1 + \cdots + u_{1(16)}\psi_{(16)}\right) + A_i\left(u_{11}\psi_1 + \cdots + u_{1(16)}\psi_{(16)}\right) \\ \frac{\partial}{\partial x_i}\left(u_{21}\psi_1 + \cdots + u_{2(16)}\psi_{(16)}\right) + A_i\left(u_{21}\psi_1 + \cdots + u_{2(16)}\psi_{(16)}\right) \\ \vdots \\ \frac{\partial}{\partial x_i}\left(u_{(16)1}\psi_1 + \cdots + u_{(16)(16)}\psi_{(16)}\right) + A_i\left(u_{(16)1}\psi_1 + \cdots + u_{(16)(16)}\psi_{(16)}\right) \end{bmatrix}$$

$$= \begin{bmatrix} u_{11}\frac{\partial_1\psi_1}{\partial x_i} + \cdots + u_{1(16)}\frac{\partial_1\psi_{(16)}}{\partial x_i} + u_{11}A_i\psi_1 + \cdots + u_{1(16)}A_i\psi_{(16)} \\ u_{21}\frac{\partial_1\psi_1}{\partial x_i} + \cdots + u_{2(16)}\frac{\partial_1\psi_{(16)}}{\partial x_i} + u_{21}A_i\psi_1 + \cdots + u_{2(16)}A_i\psi_{(16)} \\ \vdots \\ u_{(16)1}\frac{\partial_1\psi_1}{\partial x_i} + \cdots + u_{(16)(16)}\frac{\partial_1\psi_{(16)}}{\partial x_i} + u_{(16)1}A_i\psi_1 + \cdots + u_{(16)(16)}A_i\psi_{(16)} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1(16)} \\ u_{21} & u_{22} & \cdots & u_{2(16)} \\ \vdots & \vdots & \ddots & \vdots \\ u_{(16)1} & u_{(16)2} & \cdots & u_{(16)(16)} \end{bmatrix} \begin{bmatrix} \frac{\partial_1\psi_1}{\partial x_i} + A_i\psi_1 \\ \frac{\partial_1\psi_2}{\partial x_i} + A_i\psi_2 \\ \vdots \\ \frac{\partial_1\psi_{(16)1}}{\partial x_i} + A_i\psi_{(16)} \end{bmatrix}$$

$$= U\nabla_i(\Psi)$$

For the second equation:

$$\begin{aligned} \rho_2^+(F_A) &= U\left(\rho_1^+(\eta)\right)U^* \text{ (from Lemma)} \\ &= U\left(\Psi\Psi^*\right)_0 U^* \text{ (since } \Psi \text{ is a solution)} \\ &= \left((U\Psi)(U\Psi)^*\right)_0 \text{ (from Lemma)} \end{aligned}$$

To summarise, we can express the following theorem:

Theorem 3.6. Let (Γ, W) be any spin^c-structure on $\mathbf{R}^8$. Then there are no nontrivial solutions of the Seiberg-Witten equations on $\mathbf{R}^8$ with arbitrary spin^c-structure, i. e. $\rho^+(F_A) = (\Psi\Psi^*)_0$ implies $F_A = 0$ and $\Psi = 0$.

Proof. Let (A, Ψ) be a solution to the Seiberg-Witten equations on $\mathbf{R}^8$ with respect to (Γ, W). Since standard spin^c-structure is equivalent to the any spin^c-structure (Γ, W), there exists a unitary isomorphism U which satisfies the equation (2). Then the pair $(A, U\Psi)$ is a solution for the Seiberg-Witten equations on $\mathbf{R}^8$ with respect to standard spin^c-structure and from Theorem 3.1., $A = 0$ and $U\Psi = 0$. Since U is a isomorphism we get $\Psi = 0$.

ÖZET

Salamon'un genelleştirdiği Seiberg-Witten denklemleri herhangi bir çift boyutta anlamlıdır. Bu çalışmada $\mathbf{R}^8$ üzerindeki herhangi bir spin^c yapısı için Salamon tarafından verilen Seiberg-Witten denklemlerinin nontrivial çözümünün olmadığı gösterilmiştir.

REFERENCES

[1] BILGE,A. H., DERELI,T., KOÇAK, Ş., *Seiberg-Witten equations on* $\mathbf{R}^8$, Proceedings of 5. Gökova Geometry-Topology Conference,(1996), 87-92.

[2] BILGE,A. H., DERELI,T., KOÇAK, Ş., *Monopole equations on 8-manifolds with spin(7) holonomy*, Communications in Mathematical Physics **203(1)**, 1999.

[3] GAO,Y. H., TIAN, G., *Instantons and the monopole-like equations in eight dimensions*, Journal of High Energy Physics, **(5)**, 2000.

[4] LAWSON,H. B. and MICHELSOHN, M-L, *Spin Geometry*, Princeton University Press, 1989.

[5] MORGAN, J. W., *Seiberg-Witten Equations and Topology of Smooth Four Manifolds*, Princeton Univ. Press, 1996.

[6] NABER, G. L., *Topology, Geometry, and Gauge Fields (Interactions)*, Springer-Verlag, 2000.

[7] SALAMON, D., *Seiberg-Witten Invariants and Spin Geometry*, Preprint.