ANADOLU ÜNİVERSİTESİ BİLİM VE TEKNOLOJİ DERGİSİ
**ANADOLU UNIVERSITY JOURNAL OF SCIENCE AND TECHNOLOGY**
Cilt/Vol.: 3 - Sayı/No: 3 : 405-408 (2002)
XIV. Ulusal Matematik Sempozyumu Özel Sayısı/Special Issue for XIVth National Mathematics Symposium

ARAŞTIRMA MAKALESİ/**RESEARCH ARTICLE**

# COHEN'S THEOREM AND EAKIN-NAGATA-FORMANEK THEOREM

**Halil İbrahim KARAKAŞ**[1]

## *ABSTRACT*

The purpose of this work is to show that the classical *Cohen's Theorem* and *Eakin-Nagata-Formanek Theorem* are parts of one single theorem on modules. Here is the theorem:

**Theorem.** *For a module $M$ over a commutative ring $R$ with identity, the following statements are equivalent:*

**a)** *$MJ$ is finitely generated for each ideal $J$ of $R$.*
**b)** *$MP$ is finitely generated for each prime ideal $P$ of $R$.*
**c)** *Every prime submodule of $M$ is finitely generated.*
**d)** *$M$ is Noetherian.*
**e)** *$M$ satisfies the maximum condition on extended submodules.*
**f)** *$M$ satisfies the ascending chain condition on extended submodules.*

Equivalence of **c)** and **d)** implies classical *Cohen's Theorem* while equivalence of **d)** and **e)** implies *Eakin-Nagata-Formanek Theorem.*

**Key Words:** Noetherian modules, Prime submodules, Extended submodules

# COHEN TEOREMİ VE EAKİN-NAGATA-FORMANEK TEOREMİ

## *ÖZ*

Bu çalışmanın amacı, klasik olarak *Cohen Teoremi* ve *Eakin-Nagata-Formanek Teoremi* olarak bilinen teoremlerin bir tek teoremin parçaları olduğunu sergilemektir. Söz konusu teoremlerin modüllere bir genellemesi olan bu teorem şöyle ifade edilebilir:

**Teorem.** Birimli değişmeli bir $R$ halkası üzerinde bir $M$ modülü (sağ modül) için aşağıdaki önermeler denktir:

**a)** *$R$ nin her ideali $J$ için $MJ$ sonlu üretilmiştir.*
**b)** *$R$ nin her asal ideali $P$ için $MP$ sonlu üretilmiştir.*
**c)** *$M$ nin her asal altmodülü sonlu üretilmiştir.*
**d)** *$M$ Noetheryendir.*
**e)** *$M$ , genişletilmiş altmodülleri üzerinde maksimum koşulunu sağlar.*
**f)** *$M$ , genişletilmiş altmodülleri üzerinde artan zincir koşulunu sağlar.*

Bu önermelerden **c)** ve **d)** nin denkliği klasik *Cohen Teoremi*ni, **e)** ve **d** ) nin denkliği ise *Eakin-Nagata-Formanek Teoremi*ni verir.

**Anahtar Kelimeler:** Noetheryen modüller, Asal altmodüller, Genişletilmiş altmodüller

[1]Başkent Üniversitesi Uygulamalı Bilimler Yüksek Okulu Ankara

**Received:** 15 February 2002; **Revised:** 18 September 2002; **Accepted:** 25 Novamber 2002.

## 1. INTRODUCTION

The purpose of this work is to show that the classical *Cohen's Theorem* and *Eakin-Nagata-Formanek Theorem* are parts of one single theorem on modules.

For Eakin-Nagata-Formanek Theorem we refer to Eakin (1968), Nagata (1968, 1992, 1993), Formanek (1973), Smith (1981) and Jothilingam (2000). This theorem states that if a finitely generated module over a commutative ring satisfies the maximum condition for extended modules, then it is Noetherian.

Cohen's Theorem states that a commutative ring is Noetherian if and only if all its prime ideals are finitely generated. A generalization of Cohen's Theorem for modules was first given in Karakas (1972). Later Lu (1984), Smith (1981) and Jothilingam (2000) gave generalizations some of which involve extended submodules. Those generalizations of Cohen's theorem which involve extended submodules have close connection with Eakin-Nagata-Formanek Theorem. Michler (1972) gave a non-commutative version of Cohen's theorem for rings.

We will gather all these results in one single theorem and give a unified proof.

## 2. NOTATIONS, TERMINOLOGY AND DEFINITIONS

By a *ring* we will always mean a commutative ring with multiplicative identity 1, and by a *module* over a ring, we will mean a unitary right module. $A \subset B$ will denote that $A$ is a proper subset of $B$.

Let $R$ be a ring and let $M$ be an $R-$module. For a submodule $N$ of $M$, an element $x \in M$ and an element $s \in R$, we define

$$(N : M) = \{r \in R : Mr \subset N\},$$

$$N(s) = \{x \in M : xs \in N\},$$

$$(N : x) = \{r \in R : xr \in N\}.$$

It is not difficult to see that $(N : M)$ is an ideal ( a two sided ideal) of $R$; actually, it is the annihilator of the factor module $M/N$; $N(s)$ is a submodule of $M$ containing $N$; and $(N : x)$ is a right ideal of $R$ containing $(N : M)$.

A submodule $N$ of an $R-$module $M$ is said to be an *extended submodule* if $N = MJ$ for some ideal $J$ of $R$.

A submodule $N$ of an $R-$module $M$ is said to be *prime* if $N \neq M$ and if whenever $xt \in N$ for $x \in M$ and $t \in R$, one has $x \in N$ or $t \in (N : M)$. An ideal of $R$ is said to be a *prime ideal* if it is prime when it is regarded as a submodule of the right $R-$module $R$. This coincides with the usual definition of a prime ideal in a commutative ring with identity. One can also show that if $N$ is a prime submodule, then $(N : x)$ is a prime ideal of $R$ for any $x \in M\backslash N$.

It is easily seen that if $N$ is a prime submodule of an $R-$module $M$, then $(N : M)$ is a prime ideal ( two sided prime ideal) of $R$.

Equivalent definitions exist in the literature for the concept of prime submodules. For example, Dauns(1978), Smith(1981) and Lu(1984) contain alternative definitions and they discuss properties of prime submodules.

The $R-$module $M$ is said to be *Noetherian* if every submodule of $M$ is finitely generated. The ring $R$ is said to be *Noetherian* if every ideal of $R$ is finitely generated ; i.e., if $R$ is Noetherian as a right $R-$module.

## 3. COHEN-EAKIN-NAGATA-FORMANEK THEOREM

The following three lemmas will be crucial for what follows. The first lemma can be found in Karakas(1972). We include it here for completeness.

**Lemma 1.** *Let $N$ be a submodule of an $R-$module $M$. If there exists an element $s \in R\backslash(N : M)$ such that $N(s)$ and $N + Ms$ are both finitely generated, then $N$ itself is finitely generated.*

*Proof.* For an element $s \in R\backslash(N : M)$, consider the natural $R-$endomorphism $\Phi_s : M \longmapsto M$ defined by $\Phi_s(x) = xs$ for each $x \in M$. Note that

$$N(s) = \Phi_s^{-1}(N \cap Ms),$$

and thus $\Phi_s(N(s)) = N \cap Ms$. If $N(s)$ is finitely generated, its homomorphic image $N \cap Ms$ is also finitely generated. Similarly, if $N + Ms$ is finitely generated, then the factor module $(N + Ms)/Ms$ is finitely generated, too. This factor module is isomorphic to $N/(N \cap Ms)$ which is then finitely generated as an $R-$module. $N \cap Ms$ and $N/(N \cap Ms)$ being finitely generated, $N$ is finitely generated.

We note that the statement of Lemma 1 is trivially true if $s \in (N : M)$. For, $N + Ms = N$ in that case.

**Lemma 2.** *If $M$ is a finitely generated $R-$module and if $N$ is a submodule which is maximal among all nonfinitely generated submodules of $M$, then $N$ is a prime submodule of $M$ and thus $(N : M)$ is a prime ideal of $R$.*

*Proof.* Clearly, $N \neq M$. Suppose $N$ is not prime. Then there exist elements $x \in M\backslash N$ and $s \in R\backslash(N : M)$ such that $xs \in N$. In this situation, $N(s)$ and $N+Ms$ are submodules both properly containing $N$. Since $N$ is maximal among all nonfinitely generated submodules of $M$, the submodules $N(s)$ and $N+Ms$

must be finitely generated. But then Lemma 1 implies that $N$ itself must be finitely generated. This contradiction proves that $N$ is a prime submodule of $M$ and therefore $(N : M)$ is a prime ideal of $R$.

**Lemma 3.** *A finitely generated $R-$module $M$ is Noetherian if and only if $MP$ is finitely generated for each prime ideal $P$ of $R$.*

*Proof.* Since the "only if" part is trivial, we shall prove the "if" part.

Suppose that $M$ is not Noetherian. Let $N$ be a submodule which is maximal among all nonfinitely generated submodules of $M$(Such a submodule exists by Zorn's Lemma). Put $P = (N : M)$. Then $N$ is a prime submodule of $M$ and $P$ is a prime ideal of $R$ by Lemma 2. Thus $MP \subset N \subset M$ by the hypothesis. Take an element $x \in M\backslash N$ and put $J = (N : x)$. There are two possibilities: $P = J$ or $P \subset J$. If $P = J$, we consider the submodule $N + xR$. By our maximality assumption on $N$, the submodule $N + xR$ is finitely generated. Let $\{n_i + xt_i \ : \ 1 \leq i \leq p\}$ be a set of generators for $N + xR$. Here $n_i \in N$ and $t_i \in R$ for all $1 \leq i \leq p$. Thus, each $z \in N$ can be expressed as

$$z = \sum_{i=1}^{p}(n_i + xt_i)r_i = \sum_{i=1}^{p} n_i r_i + x(\sum_{i=1}^{p} t_i r_i)$$

with $r_i \in R$ for each $1 \leq i \leq p$. This expression shows that $x(\sum_{i=1}^{p} t_i r_i) \in N$ and thus $(\sum_{i=1}^{p} t_i r_i) \in J = P$. Hence

$$N \subseteq n_1R + n_2R + ... + n_pR + xP.$$

On the other hand, we know that $xP \subset N$; therefore the last inclusion is actually an equality:

$$N = n_1R + n_2R + ... + n_pR + xP.$$

Taking into consideration the fact that $xP \subseteq MP \subset N$, we obtain

$$\begin{aligned} N = \ & n_1R + n_2R + ... + n_pR + xP \\ & \subseteq n_1R + n_2R + ... + n_pR + MP \\ & \subseteq n_1R + n_2R + ... + n_pR + N \subseteq N. \end{aligned}$$

It follows that

$$N = n_1R + n_2R + ... + n_pR + MP.$$

Since $MP$ is finitely generated, the last identity shows that $N$ is finitely generated; which is a contradiction. Now, let us consider the case $P \subset J$. In this case, there exists an element $s \in R\backslash P$ such that $xs \in N$. Thus $N$ is properly contained in both $N(s)$ and $N+Ms$. Our assumption about $N$ implies that $N(s)$ and $N + Ms$ are both finitely generated. But then Lemma 1 implies that $N$ itself is finitely generated; which is a contradiction. This proves the assertion.

**Theorem 1.** *For a finitely generated $R-$module $M$, the following statements are equivalent:*

**a)** *$MJ$ is finitely generated for each ideal $J$ of $R$.*

**b)** *$MP$ is finitely generated for each prime ideal $P$ of $R$.*

**c)** *Every prime submodule of $M$ is finitely generated.*

**d)** *$M$ is Noetherian.*

**e)** *$M$ satisfies the maximum condition on extended submodules.*

**f)** *$M$ satisfies the ascending chain condition on extended submodules.*

**Remark.** Equivalence of **c)** and **d)** implies classical *Cohen's Theorem* while equivalence of **d)** and **e)** implies *Eakin-Nagata-Formanek Theorem.*

*Proof.* The implications **a)** $\Longrightarrow$ **b)**, **d)** $\Longrightarrow$ **e) and** and **e)** $\Longrightarrow$ **f)**are obvious. **b)** $\Longrightarrow$ **c)** by Lemma 3 and **c)** $\Longrightarrow$ **d)** by Lemma 2 . Thus, it remains to prove the implication **f)** $\Longrightarrow$ **a)**. Suppose that $MJ$ is not finitely generated for ideal $J$ of $R$. Take an element $t_1 \in J$. Let $J_1 = t_1R$, the ideal generated by $t_1$ in $R$. Since $M$ is finitely generated, $MJ_1$ is finitely generated. Therefore, $MJ_1 \neq MJ$. Thus there exists $t_2 \in J$ such that $Mt_2 \not\subseteq MJ_1$. Let $J_2 = t_1R + t_2R$, the ideal generated by $\{t_1, t_2\}$ in $R$. $MJ_2$ is finitely generated and we have

$$MJ_1 \subset MJ_2 \subset MJ.$$

There exists $t_3 \in J$ such that $Mt_3 \not\subseteq MJ_2$. Let $J_3 = t_1R + t_2R + t_3R$. Then $MJ_3$ is finitely generated and we have the strictly ascending chain of submodules

$$MJ_1 \subset MJ_2 \subset MJ_3 \subset MJ.$$

Continuing this process, we find a sequence of elements $t_1, t_2, ..., t_n, ...$ of $J$ such that with $J_n = t_1R + t_2R + ... + t_nR$, the chain

$$MJ_1 \subset MJ_2 \subset ... \subset MJ_n \subset ...$$

turns out to be an infinite strictly ascending chain of extended submodules of $M$. This is a contradiction since $M$ satisfies the ascending chain condition on extended submodules. Hence $MJ$ is finitely generated, proving **f)** $\Longrightarrow$ **a)**.

Equivalence of **c)** and **d)** in Theorem 1 generalizes the classical Cohen's Theorem while the equivalence of **e)** and **a)** generalizes Eakin-Nagata-Formanek Theorem:

**Corollary 1(Cohen).** *A commutative ring with identity is Noetherian if and only if all its prime ideals are finitely generated.*

**Corollary 2(Eakin-Nagata-Formanek).** *Let $R$ be a commutative ring and let $M$ be a finitely generated $R-$module. If $M$ satisfies the maximum condition on extended submodules, then $M$ is a noetherian $R-$module.*

## REFERENCES


Cohen, I. S., (1950). Commutative rings with restricted minimum condition. *Duke Math. J.* 17, 27–42.

Dauns, J. (1978). Prime Modules. *J. reine angew. Math.* 298, 156–181.

Eakin, P. M. (1968). The converse to a well known theorem on noetherian rings. *Math. Ann.* 177, 278–282.

Formanek, E. (1973). Faithful noetherian modules. *Proc. Amer. Math. Soc.* 41, 381–383.

Jothlingam, P. (2000). Cohen's Theorem and Eakin-Nagata-Formanek Theorem revisited. *Comm. in Alg.* 28, 4861–4866.

Karakas, H. I. (1972). On neotherian modules. *METU J. Pure Appl. Sci. Soc.* 5, 165–168.

Lu, C. P. (1984). Prime submodules of modules. *Comment. Math. Univ. Sancti Pauli* 33, 61–69.

Michler, G. O. (1972). *Prime right ideals and right neotherian rings.* in ring theory(Proc. Conf. Park city, Utah), Academic Press, New York, 61–69.

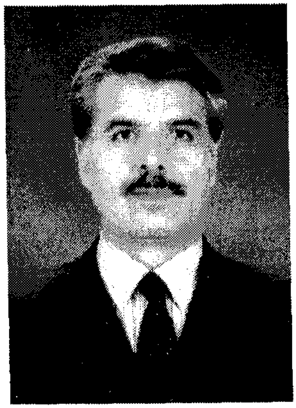
Nagata, M. (1968). A type of subring of a neotherian ring. *J. Math. Kyoto Univ.* 8, 465–467.

Nagata, M. (1992). A new proof of the theorem of Eakin - Nagata. *Chin. Math.* 20, 1–3.

Nagata, M. (1993). On Eakin-Nagata-Formanek Theorem. *J. Math. Kyoto Univ.* 33, 825–826.

Smith, P. F. (1981). The injective test lemma in fully bounded rings. *Comm. in Alg.* 9, 1701–1708.

Smith, P. F. (1981). *Concerning a Theorem of I.S. Cohen.* manuscript.


**Halil İbrahim KARAKAŞ**

1946 yılında Antalya'da doğdu. 1968 yılında ODTÜ Matematik bölümünü bitirdi. 1970 yılında yüksek lisans, 1974 yılında doktora çalışmalarını tamamladı. Halen Başkent Üniversitesi'nde Bankacılık bölümü başkanı olarak görev yapmaktadır.