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OPTIMIZATION OF LOSS PROBABILITY FOR $GI/M/3/0$ QUEUING SYSTEM WITH HETEROGENOUS SERVERS

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ABSTRACT

In this paper a queuing system with recurrent arrivals, three heterogeneous servers, and no waiting line is examined. In this system an arriving customer may choose any one of the free servers with equal probability. When all servers are busy, customers beyond the capacity of the system are lost. These customers are called "lost customers". The probability of losing a customer is computed for the queuing system, and it is shown that when the mean of the interarrival time distribution is fixed, loss probability is minimized by deterministic interarrival time distribution. This conclusion is supported by the simulation results.

Keywords: Deterministic distribution, Heterogeneous servers, Loss probability, LS transform, Stream of overflow, Optimization.

HETEROJEN KANALLI $GI/M/3/0$ KUYRUK SİSTEMİNDE KAYBOLMA OLASILIĞININ OPTİMİZASYONU

ÖZ

Bu çalışmada rekurrent girişli, bekleme yerinin olmadığı, 3 heterojen kanallı bir kuyruk modeli incelenmiştir. Bu sistemde gelen müşteri, boş olan kanallardan herhangi birisine eşit olasılıkla girer. Bütün kanallar dolu ise sistem kapasitesi aşıldığından, gelen müşteri hizmet almadan sistemden ayrılır. Bu tür müşterilere "kaybolan müşteri" denir. İncelenen kuyruk sisteminde müşterinin kaybolma olasılığı hesaplanmış ve ortalaması sabit olan gelişlerarası süre dağılımları içinden gelişlerarası süre dağılımı deterministik seçildiğilde kaybolma olasılığının minimum olduğu gösterilmiştir. Elde edilen sonuçlar bir simülasyon çalışmasıyla desteklenmiştir.

Anahtar Kelimeler: Deterministik dağılım, Heterojen servis birimi, Kaybolma olasılığı, LS dönüşümü, Kaybolan müşteri akımı, Optimizasyon.

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1. INTRODUCTION

The $M/M/n/0$ queuing model, known as Erlang's loss model, was first analyzed by Erlang **[4]**, and the stationary probability that k servers of the system are occupied is given by Erlang's loss formula:

$$P_k = \frac{(\lambda/\mu)^k/k!}{\sum_{k=0}^{n}\frac{(\lambda/\mu)^k}{k!}}, \qquad 0 \le \mathrm{k} \le n \tag{1.1}$$

In Erlang's formula, $1/\lambda$ is the mean interarrival time, and $1/\mu$ is the mean service time. Palm **[10]** indicates that the stream of the lost customers from a $GI/M/n/0$ queuing system forms a renewal process and derives the Laplace-Stieltjes (LS) transform for the times between customer losses. Palm **[10]** gives the loss probability for the the $GI/M/n/0$ queuing model as in (1.2):

$$\frac{1}{P_n} = \sum_{k=0}^{n} \binom{n}{k} c_k \tag{1.2}$$

where k is the number of busy servers in the system and c_k is

$$c_0 = 1\ ,\ c_k = \frac{1-f(\mu)}{f(\mu)} \cdots \frac{1-f(k\mu)}{f(k\mu)}\ ,\ k \ge 1 \tag{1.3}$$

In (1.3), f is the LS transform of the interarrival time distribution. Erlang's formula was extended to the case of dependent service times by Konig and Matthes **[7]**. Takacs **[16]** analyzed the model M/M/n/0, which was introduced initially by Erlang (1917), considering the discrete parameter stochastic process (Markov chains) to describe arrival and departure times. Brumelle **[2]** generalized Erlang's loss system to state dependent arrival and service rates. Halfin **[6]** derived distribution of interoverflow times in the model $GI/G/1$ with no waiting line.

Generally in queuing models, a restrictive assumption is made about the servers' homogeneity, that is the mean service time is identical for all servers. In reality, the mean service time is not identical for all servers. Because of the growing abundance of technology such as automated telephone systems and production lines, there is a need for continual analysis refinement of queuing models.
Gumbel **[5]** and Blanc **[1]** obtained the limiting distribution of the number of customers for the $M/M/n$ queuing system with heterogeneous exponential servers with the assumption that the queue length is unbounded. Singh **[14]** examined the Markovian queuing system $M/M_i/2/(\beta)$ with two heterogeneous servers. Besides he calculated the mean of queue length, the mean holding time and the mean number of customers in the system. Then he compared the results with the model $M/M/2/(\beta)$ with two homogenous servers. Singh **[15]** obtained the average characteristics of Markovian queuing system $M/M_i/3$ with heterogeneous servers. Nath and Ens **[9]** proved that with the fastest-service rule, loss probability was minimum from the queuing model $M/M/n/0$ with heterogeneous servers. Kumar, Madheswari and Venkatakrishan **[8]** examined the Markovian queuing system $M/M/2$ with heterogeneous servers and catastrophes, besides they calculated the average characteristics of the system.

Our paper mainly builds on the results of Palm's paper (1943); however, unlike the existing literature, we analyze a model with three heterogeneous service channels. The $GI/M/3/0$ queuing model with heterogeneous servers is examined. In addition to the calculation of customer loss probability, minimization of the loss probability is examined. In the second part of the paper, model formulation and related assumptions are given. The details of the simulation study are explained, and the results are proposed in the following section. Finally, conclusions and future research directions are presented in the last section.

2. MODEL FORMULATION

We consider the queuing system in which the arrival process may have any general distribution whereas the service process is exponential. The interarrival times are independent and identically distributed random variables whose distribution function is $F(t)$. There are three service channels whose mean service times are assumed to be different. The service time for each customer getting service from the *k*th service channel is an exponential random variable with parameter μ_k $(k = 1,2,3)$ and represented by η_k.

$$P(\eta_k \le t) = 1 - e^{-\mu_k t} \quad , \ t \ge 0 \tag{2.1}$$

The service discipline is assumed to be "random" in the sense that the probability of an arriving customer getting service from any idle server is equal. Provided that all the servers are busy, the customer cannot be served, and that customer is lost.

The defined queuing system is represented as $GI/\vec{M}/3/0$. In this setting $\vec{M}$ indicates that the service channels are not homogeneous. That is, the parameters of the distribution functions are different from each other.

Let $\tau_1 < \tau_2 < \dots$ be arrival times to the system when all servers are busy. The sequence $\{\tau_n\}$ is called the stream of overflow. The sequence $\{\tau_n\}$ is a renewal process i.e. $\tau_2 - \tau_1, \tau_3 - \tau_2, \dots$ are independent, identically distributed random variables, τ_1 and $\{\tau_k - \tau_{k-1}, \, k \ge 2\}$ are independent. The aim of this study is to examine the stream of overflow for the $GI/\vec{M}/3/0$ queuing model and to find the function F($\int_0^\infty [1 - F(t)]dt = a$ where a is constant) that minimizes the loss probability. The problem of describing the stream of overflow for a finite queue with a recurrent arrival and single negative exponential server is considered by Çinlar and Disney **[3]**.

2.1 Semi-Markov Process Representing the System

Let $X(t)$ denote the customer number in the system at time t; in particular let $X_n = X(t_n - 0)$, $n \ge 1$. If a customer arrives and finds that all servers are occupied, she/he departs never to return and is said to have overflowed.

We define a semi-Markov process $\{\xi(t), \, t \ge 0\}$ as $\xi(t) = X_n$, if and only if $t_n \le t < t_{n+1}$. We denote by $Q_{ij}(x)$ the kernel of process $\xi(t)$, that is

$$Q_{ij}(x) = P\{X_{n+1} = j \, , \, t_{n+1} - t_n \le x \big| X_n = i\} \, , \tag{2.2}$$

where $x \ge 0$ and $0 \le i, j \le 3$. These $Q_{ij}(x)$ states can be obtained for the process as follows:

$$Q_{00}(x) = F(x) - Q_{01}(x) \ ,$$

$$Q_{01}(x) = \frac{1}{3} \int_0^x (e^{-\mu_1 t} + e^{-\mu_2 t} + e^{-\mu_3 t}) \, dF(t) \, ,$$

$Q_{02}(x)=0$, $Q_{03}(x)=0$,

$$Q_{10}(x)=\frac{1}{3}\int_0^x [3-2e^{-\mu_1 t}-2e^{-\mu_2 t}-2e^{-\mu_3 t}+e^{-(\mu_1+\mu_2)t}+e^{-(\mu_1+\mu_3)t}+e^{-(\mu_2+\mu_3)t}]dF(t) \ ,$$

$$Q_{11}(x)=F(x)-Q_{10}(x)-Q_{12}(x),$$

$$Q_{12}(x)=\frac{1}{3}\int_0^x (e^{-(\mu_1+\mu_2)t}+e^{-(\mu_1+\mu_3)t}+e^{-(\mu_2+\mu_3)t})\,dF(t),\ \ Q_{13}(x)=0\,,$$

$$Q_{20}(x)=\int_0^x [1-e^{-\mu_1 t}-e^{-\mu_2 t}-e^{-\mu_3 t}+e^{-(\mu_1+\mu_2)t}+e^{-(\mu_1+\mu_3)t}+e^{-(\mu_2+\mu_3)t}-e^{-(\mu_1+\mu_2+\mu_3)t}]dF(t),$$

$$Q_{21}(x)=\int_0^x [e^{-\mu_1 t}+e^{-\mu_2 t}+e^{-\mu_3 t}-2(e^{-(\mu_1+\mu_2)t}+e^{-(\mu_1+\mu_3)t}+e^{-(\mu_2+\mu_3)t})+3e^{-(\mu_1+\mu_2+\mu_3)t}]dF(t),$$

$$Q_{22}=F(x)-Q_{20}(x)-Q_{21}(x)-Q_{23}(x),$$

$$Q_{23}(x)=\int_0^x e^{-(\mu_1+\mu_2+\mu_3)t}\,dF(t),$$

$$Q_{3j}(x)=Q_{2j}(x), \quad j=0,1,2,3.$$

Let $q_{ij}(s)$ be the LS transform of $Q_{ij}(x)$, and $f(s)$ be the LS transform of $F(x)$. The $q(s)$ is the square matrix of elements $q_{ij}(s)$. The $q(s)=[q_{ij}(s)]_0^3$ matrix can be expressed as in (2.3).

$$q(s)=\begin{bmatrix} f(s)-\frac{1}{3}f_1(s) & \frac{1}{3}f_1(s) & 0 & 0 \\ f(s)-\frac{2}{3}f_1(s)+\frac{1}{3}f_2(s) & \frac{2}{3}[f_1(s)-f_2(s)] & \frac{1}{3}f_2(s) & 0 \\ f(s)-f_1(s)+f_2(s)-f_3(s) & f_1(s)-2f_2(s)+3f_3(s) & f_2(s)-3f_3(s) & f_3(s) \\ f(s)-f_1(s)+f_2(s)-f_3(s) & f_1(s)-2f_2(s)+3f_3(s) & f_2(s)-3f_3(s) & f_3(s) \end{bmatrix} \tag{2.3}$$

where

$$\begin{aligned} f_1(s)&=f(s+\mu_1)+f(s+\mu_2)+f(s+\mu_3), \\ f_2(s)&=f(s+\mu_1+\mu_2)+f(s+\mu_1+\mu_3)+f(s+\mu_2+\mu_3), \\ f_3(s)&=f(s+\mu_1+\mu_2+\mu_3) \end{aligned} \tag{2.4}$$

Now we suppose that $p_{ij}=P(X_n=j\,/\,X_{n+1}=i)$ is the one-step transition probability of $\{X_n\}$. Since $p_{ij}=q_{ij}(0)$, the matrix P may be obtained as follows:

$$P = \begin{bmatrix} 1-\frac{1}{3}f_1 & \frac{1}{3}f_1 & 0 & 0 \\ 1-\frac{2}{3}f_1+\frac{1}{3}f_2 & \frac{2}{3}(f_1-f_2) & \frac{1}{3}f_2 & 0 \\ 1-f_1+f_2-f_3 & f_1-2f_2+3f_3 & f_2-3f_3 & f_3 \\ 1-f_1+f_2-f_3 & f_1-2f_2+3f_3 & f_2-3f_3 & f_3 \end{bmatrix} \tag{2.5}$$

where $f_k = f_k(0)$, $k = 1,2,3$.

2.2 The Analysis of Overflow Process

Let $\{\xi(t),\ t \ge 0\}$ be the semi-Markov process whose state space is $S = \{0,1,\dots,n\}$ and suppose $Q_{ij}(t) = p_{ij}F_{ij}(t)$ is the kernel of $\xi(t)$ where p_{ij} is the one-step transition probability for the embeded Markov Chain, and F_{ij} is the distribution function of the sojourn time in state i, given that the next state is j .

Let T_{0n} be first passage time to go from state 0 to *n*, T_{nn} be the recurrence time to state *n*. In addition, let $\varphi_{0n}(s)$, $\varphi_{nn}(s)$ and $q_{ij}(s)$ be the LS transforms of T_{0n}, T_{nn} and $Q_{ij}(x)$ respectively.

Pyke **[11-12]** proved that inverse of the matrix $I - q(s) = (\delta_{ij} - q_{ij}(s))$ is available on the condition that $\mathrm{Re}\, s > 0$ and obtained the following formulas for φ_{0n} and φ_{nn}

$$\varphi_{0n}(s) = \Gamma_{0n}(s) / \Gamma_{nn}(s)$$

$$1 - \varphi_{nn}(s) = 1 / \Gamma_{nn}(s)$$

where $\Gamma_{ij}(s)$ is the (i,j) entry of the matrix $(I - q(s))^{-1}$ $(I - q(s))^{-1}$. Then (2.6) and (2.7) can be obtained

$$\varphi_{0,n}(s) = D_{0n}(s) / D_{nn}(s) \tag{2.6}$$

$$1 - \varphi_{n,n}(s) = |I - q(s)| / D_{nn}(s) \tag{2.7}$$

where $D_{nn}(s)$ and $D_{0n}(s)$ are the cofactors of (n,n) and $(0,n)$ entries of the matrix $I - q(s)$, respectively. The mean recurrence time to the state n can be obtained as in (2.8) by using (2.7)

$$ET_{nn} = D(m_0,\dots,m_n) / D_{nn}(0) \tag{2.8}$$

where $m_i\ (i = 0,\dots,n)$ refers to the expected value of the sojourn time in state i , $D(m_0,\dots,m_n)$ is the determinant $|I - q(0)|$ $|I - q(0)|$. The determinant $D(m_0,\dots,m_n)$ is obtained by writing $(m_0,\dots,m_n)'$ instead of the 0^{th} column in the determinant $|I - q(0)|$.

We apply the (2.6) and (2.7) formulas for the matrix $q(s)$ and obtain (2.9).

$$|I - q(s)| = (1 - f)[(1 - f + q_{10})(1 - q_{22} - q_{23}) + (1 - f + q_{20})q_{12} + q_{01}(1 - q_{22} - q_{23} + q_{12})] \tag{2.9}$$

The cofactors of $(3,3)$ and $(0,3)$ entries of the matrix $I-q(s)$ are obtained as in (2.10) and (2.11) respectively.

$$D_{33}=(1-q_{00})(1-q_{11}-q_{22}+q_{11}q_{22}-q_{12}q_{21})-q_{01}(q_{10}+q_{12}q_{20}-q_{10}q_{22}), \tag{2.10}$$

$$D_{03}=q_{01}q_{12}q_{23} \tag{2.11}$$

For $n=3$, by using (2.6) and (2.7),

$$\varphi_{03}(s)=\frac{q_{01}q_{12}q_{23}}{(1-q_{00})(1-q_{11}-q_{22}+q_{11}q_{22}-q_{12}q_{21})-q_{01}(q_{10}+q_{12}q_{20}-q_{10}q_{22})} \tag{2.12}$$

and

$$1-\varphi_{33}(s)=\frac{(1-f)[(1-f+q_{10})(1-q_{22}-q_{23})+(1-f+q_{20})q_{12}+q_{01}(1-q_{22}-q_{23}+q_{12})]}{(1-q_{00})(1-q_{11}-q_{22}+q_{11}q_{22}-q_{12}q_{21})-q_{01}(q_{10}+q_{12}q_{20}-q_{10}q_{22})} \tag{2.13}$$

can be obtained where $f=f(s)$, and $q_{ij}=q_{ij}(s)$. The formulas (2.12) and (2.13) define the stream of overflows in the $GI/\vec{M}/3/0$ queuing system.

2.3 Steady State Analysis

Using (2.6), (2.7), when $m_0=m_1=m_2=m_3=a$, the following is obtained

$$ET_{00}=\frac{\Delta}{D_{00}(0)}\ ,\quad ET_{33}=\frac{\Delta}{D_{33}(0)}\ ,\quad ET_{01}=\frac{\Delta}{D_{01}(0)}\ ,\quad ET_{02}=\frac{\Delta}{D_{02}(0)}\,,$$

$$\int_0^\infty [1-F(t)]dt=a$$

where

$$\Delta=\begin{vmatrix} a & -p_{01} & 0 & 0 \\ a & 1-p_{11} & -p_{12} & 0 \\ a & -p_{21} & 1-p_{22} & -p_{23} \\ a & -p_{21} & -p_{22} & 1-p_{23} \end{vmatrix}\ ,$$

$$\Delta=a[p_{10}(1-p_{22}-p_{23})+p_{01}(1-p_{22}-p_{23}+p_{12})+p_{12}p_{20}].$$

$D_{00}(0)$, $D_{33}(0)$, $D_{01}(0)$ and $D_{02}(0)$ are the cofactors of $(0,0)$, $(3,3)$, $(0,1)$ and $(0,2)$ entries of the matrix

$$I-P=I-q(0)=\begin{bmatrix} 1-p_{00} & -p_{01} & 0 & 0 \\ -p_{10} & 1-p_{11} & -p_{12} & 0 \\ -p_{20} & -p_{21} & 1-p_{22} & -p_{23} \\ -p_{20} & -p_{21} & -p_{22} & 1-p_{23} \end{bmatrix}\,,$$

That is,

$$D_{00}(0) = (1-p_{11})(1-p_{22}-p_{23}) - p_{12}p_{21}\,,$$

$$D_{01}(0) = P_{01}(1-P_{22}-P_{23})\,,$$

$$D_{02}(0) = p_{01}p_{12}(1-p_{23})\,,$$

$$D_{33}(0) = p_{01}p_{12}p_{23}\,.$$

The transition probabilities p_{ij} are defined in (2.5). The mean recurrence times to the state 0 and state 3 and the mean first passage time from state 0 to state 1 and state 2 are calculated as follows:

$$ET_{00} = \frac{a[p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}]}{(1-p_{11})(1-p_{22}-p_{23}) - p_{12}p_{21}}\,,$$

$$ET_{01} = \frac{a[p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}]}{p_{01}(1-p_{22}-p_{23})}\,,$$

$$ET_{02} = \frac{a[p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}]}{p_{01}p_{12}(1-p_{23})}\,,$$

$$ET_{33} = \frac{a[p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}]}{p_{01}p_{12}p_{23}}\,.$$

The steady state probabilities can be derived as follows:

$$\pi_0 = \frac{a}{ET_{00}} = \frac{(1-p_{11})(1-p_{22}-p_{23}) - p_{12}p_{21}}{p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}}\,,$$

$$\pi_1 = \frac{a}{ET_{01}} = \frac{p_{01}(1-p_{22}-p_{23})}{p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}}\,,$$

$$\pi_2 = \frac{a}{ET_{02}} = \frac{p_{01}p_{12}(1-p_{23})}{p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}}\,,$$

$$\pi_3 = \frac{a}{ET_{33}} = \frac{p_{01}p_{12}p_{23}}{p_{10}(1-p_{22}-p_{23}) + p_{01}(1-p_{22}-p_{23}+p_{12}) + p_{12}p_{20}} \qquad (2.14)$$

2.4 The Loss Probability and Its Minimization

The probability of a losing customer is equal to the probability of all the servers being busy where there is no waiting line. Hence, from (2.5) and (2.14), the loss probability is given as in (2.15).

$$P_L = \frac{f_1 f_2 f_3}{9\ (1-f_2+2f_3)(1-\frac{1}{3}f_1+\frac{1}{3}f_2) + \frac{1}{3}f_2(1-\frac{2}{3}f_1+f_2-f_3)} \qquad (2.15)$$

In (2.16), $f_k = f_k(0)$, $k = 1,2,3$ is obtained by setting $s = 0$ in (2.4). As a modification of (2.15), with the assumption of $\mu_1 = \mu_2 = \mu_3 = \mu$, yields Palm's loss formula [10] with n=3 as follows for the defined system:

$$\frac{1}{P_L} = 1 + 3\,\frac{1 - f(\mu)}{f(\mu)} + 3\,\frac{[1 - f(\mu)][1 - f(2\mu)]}{f(\mu)f(2\mu)} + \frac{[1 - f(\mu)][1 - f(2\mu)][1 - f(3\mu)]}{f(\mu)f(2\mu)f(3\mu)} \tag{2.16}$$

The analysis of optimization problems results in more efficient working systems. Signh **[15]** minimized numerically the average characteristics of the model $M / M_i / 3$ with heterogeneous system and compared to the corresponding homogeneous system $M / M / 3$. Nath and Enns **[9]** examined the $M / M / n / 0$ queuing model and found the loss probability was minimized, provided that the sum of the service rates is constant and arriving customers were assigned to the server with the shortest mean service time.

Let H_a indicate the class of interarrival time distributions F having a fixed mean a, and let $P_L(F)$ denote the loss probability for a defined system with an interarrival time distribution $F \in H_a$. And also, assume $A(t)$ is deterministic distribution, that is, $A(t) = 0$ for $t \le a$ and $A(t) = 1$ for $t > a$. It is obvious that $A \in H_a$ and e^{-as} is the LS transformations of $A(t)$.

Theorem. The loss probability $P_L(F)$, $F \in H_a$ is minimized by $F = A$.

Proof. (2.15) can be written as in (2.17).

$$P_L(F) = \frac{f_1 f_2}{9\phi(f)} \tag{2.17}$$

where

$$\begin{aligned}\phi(f) &= \frac{(1 - f_2 + 2f_3)(1 - \frac{1}{3}f_1 + \frac{1}{3}f_2) + \frac{1}{3}f_2(1 - \frac{2}{3}f_1 + f_2 - f_3)}{f_3} \\ &= \frac{(1 - \frac{1}{3}f_1 f_3) + (1 - \frac{2}{3}f_2)(-\frac{1}{3}f_1) + (1 - \frac{1}{3}f_3)(-\frac{1}{3}f_2)}{f_3} + 2\end{aligned}$$

The term $f_1 f_2 / 9$ can be rewritten as in (2.18) $f(s) \ge e^{-as}$ is attained from Jensen's inequality **[13]**

$$\begin{aligned}\frac{f_1 f_2}{9} &= [f(\mu_1) + f(\mu_2) + f(\mu_3)] \cdot [f(\mu_1 + \mu_2) + f(\mu_1 + \mu_3) + f(\mu_2 + \mu_3)] / 9 \\ &\ge (e^{-a\mu_1} + e^{-a\mu_2} + e^{-a\mu_3})(e^{-a(\mu_1+\mu_2)} + e^{-a(\mu_1+\mu_3)} + e^{-a(\mu_2+\mu_3)}) / 9\end{aligned} \tag{2.18}$$

$f(s) \ge e^{-as}$ is used to calculate each of the following elements $1 - f_1 f_3 / 3$, $1 - 2f_2 / 3$, $-f_1 / 3$, $1 - f_3 / 3$, $-f_2 / 3$ and f_3, as shown below:

$$1-\frac{1}{3}f_1f_3 = 1-\frac{1}{3}[f(\mu_1)+f(\mu_2)+f(\mu_3)]\cdot f(\mu_1+\mu_2+\mu_3)$$
$$\leq 1-\frac{1}{3}(e^{-a\mu_1}+e^{-a\mu_2}+e^{-a\mu_3})\cdot e^{-a(\mu_1+\mu_2+\mu_3)},$$

$$1-\frac{2}{3}f_2 = 1-\frac{2}{3}[f(\mu_1+\mu_2)+f(\mu_1+\mu_3)+f(\mu_2+\mu_3)]$$
$$\leq 1-\frac{2}{3}[e^{-a(\mu_1+\mu_2)}+e^{-a(\mu_1+\mu_3)}+e^{-a(\mu_2+\mu_3)}],$$

$$-\frac{1}{3}f_1 = -\frac{1}{3}[f(\mu_1)+f(\mu_2)+f(\mu_3)]$$
$$\leq -\frac{1}{3}(e^{-a\mu_1}+e^{-a\mu_2}+e^{-a\mu_3}),$$

$$1-\frac{1}{3}f_3 = 1-\frac{1}{3}f(\mu_1+\mu_2+\mu_3)$$
$$\leq 1-\frac{1}{3}e^{-a(\mu_1+\mu_2+\mu_3)},$$

$$-\frac{1}{3}f_2 = -\frac{1}{3}[f(\mu_1+\mu_2)+f(\mu_1+\mu_3)+f(\mu_2+\mu_3)]$$
$$\leq -\frac{1}{3}(e^{-a(\mu_1+\mu_2)}+e^{-a(\mu_1+\mu_3)}+e^{-a(\mu_2+\mu_3)}),$$

$$f_3 = f(\mu_1+\mu_2+\mu_3)$$
$$\geq e^{-a(\mu_1+\mu_2+\mu_3)}$$

For convenience, above equations can be written as follows:

$$\phi_1(e^{-as}) = 1-\frac{1}{3}(e^{-a\mu_1}+e^{-a\mu_2}+e^{-a\mu_3})\cdot e^{-a(\mu_1+\mu_2+\mu_3)}$$

$$\phi_2(e^{-as}) = 1-\frac{2}{3}(e^{-a(\mu_1+\mu_2)}+e^{-a(\mu_1+\mu_3)}+e^{-a(\mu_2+\mu_3)})$$

$$\phi_3(e^{-as}) = -\frac{1}{3}(e^{-a\mu_1}+e^{-a\mu_2}+e^{-a\mu_3})$$

$$\phi_4(e^{-as}) = 1-\frac{1}{3}e^{-a(\mu_1+\mu_2+\mu_3)}$$

$$\phi_5(e^{-as}) = -\frac{1}{3}(e^{-a(\mu_1+\mu_2)}+e^{-a(\mu_1+\mu_3)}+e^{-a(\mu_2+\mu_3)})$$

$$\phi_6(e^{-as}) = e^{-a(\mu_1+\mu_2+\mu_3)}$$

Then the $\phi(f)$ function can be written as follows:

$$\phi = \phi(f) \leq \frac{\phi_1(e^{-as}) + \phi_2(e^{-as}) \cdot \phi_3(e^{-as}) + \phi_4(e^{-as}) \cdot \phi_5(e^{-as})}{\phi_6(e^{-as})} + 2,$$

$$\phi(e^{-as}) = \frac{\phi_1(e^{-as}) + \phi_2(e^{-as}) \cdot \phi_3(e^{-as}) + \phi_4(e^{-as}) \cdot \phi_5(e^{-as})}{\phi_6(e^{-as})} + 2. \quad (2.19)$$

And finally (2.20) is obtained using the results from (2.18) and (2.19)

$$P_L(F) \geq \frac{(e^{-a\mu_1} + e^{-a\mu_2} + e^{-a\mu_3})(e^{-a(\mu_1+\mu_2)} + e^{-a(\mu_1+\mu_3)} + e^{-a(\mu_2+\mu_3)})}{9\phi(e^{-as})}$$

$$P_L(A) = \frac{(e^{-a\mu_1} + e^{-a\mu_2} + e^{-a\mu_3})(e^{-a(\mu_1+\mu_2)} + e^{-a(\mu_1+\mu_3)} + e^{-a(\mu_2+\mu_3)})}{9\phi(e^{-as})} \quad (2.20)$$

$$P_L(F) \geq P_L(A)$$

As a result, $P_L(F)$ is minimized when the interarrival time is constant with probability one.

3. SIMULATION STUDY

The queuing system with three heterogeneous servers was simulated. Three different distributions considered for the arrival process. When a customer enters the system, she is served either by server 1, server 2 or server 3, whichever becomes available first. If all of them or two of them are available at the same time, the customers are served by any server with equally likely. If all of the servers are busy, the customer is lost.

Service process was considered to be exponential and the mean service times for servers were selected arbitrarily such as 4.08, 3.15, and 4.88, for server 1, 2 and 3, respectively. It has been proved that the difference between mean service times is statistically significant for α=0.01. Since service times are exponential, the distribution assumption of ANOVA is violated. Hence, the significance test of differences between service times for each server was tested by Kruskal Wallis. The test statistics was obtained as H=74.71; when it was compared with the chi-square value (9.21) for 2 degrees of freedom and 0.01 significance level, the null hypothesis is rejected. It was concluded that each of three exponential server has different parameter.

Kruskal-Wallis Test

Kruskal-Wallis Test on Service Times				
Servers	N	Median	Ave Rank	Z
1	1000	2.733	1505.5	0.20
2	1000	2.174	1330.9	-7.58
3	1000	3.411	1665.6	7.38
Overall	3000		1500.5	
H=74.71	DF=2	P=0.000		

The simulation of the system begins with an empty system and it is arbitrarily assumed that the first event, an arrival, takes place at clock time 0. The state of the system is modified by events; in other words, discrete event simulation was used to simulate the queuing system. The events which are arrivals or departures occur at discrete time points. Associated with each event, there is a clock which indicates that event's scheduled time to occur. At the time of any customer departure, the simulation schedules the next event. If no customers in the system, no departure is scheduled and the simulation clock is set to 9999. When a customer enters to a system, the simulation clock is reset according to service time distribution.

The scope of the simulation study is to find the loss probability under the assumption of different distributions of interarrival times and to indicate that the minimum loss probability is achieved when the interarrival times are deterministic. A partial listing of results after running the simulation is given in Table 1.

Table 1. A sample output of system simulation

TM	Event	CN	Server 1	Server 2	Server 3	Number of the Lost Customers	AT	DT
0.0000	0	0	0	0	0	0	0.0000	9999.0000
0.0000	1	1	0	0	1	0	1.8600	11.0629
1.8600	1	2	1	0	1	0	1.9217	11.0629
1.9217	1	3	1	1	1	0	4.1228	7.9152
4.1228	1	4	1	1	1	1	5.5701	7.9152
5.5701	1	5	1	1	1	2	6.4621	7.9152
6.4621	1	6	1	1	1	3	7.9272	7.9152
7.9152	0	1	1	0	1	0	7.9272	11.0629
7.9272	1	7	1	1	1	0	8.1848	11.0629
8.1848	1	8	1	1	1	1	11.6520	11.0629
11.0629	0	2	1	1	0	0	11.6520	12.8413
11.6520	1	9	1	1	1	0	12.6903	12.8413
12.6903	1	10	1	1	1	1	15.0165	12.8413
12.8413	0	3	1	0	1	0	15.0165	16.1897
15.0165	1	11	1	1	1	0	15.6177	16.1897
15.6177	1	12	1	1	1	1	16.3625	16.1897
16.1897	0	4	0	1	1	0	16.3625	16.2440
16.2440	0	5	0	1	0	0	16.3625	17.9879
16.3625	1	13	0	1	1	0	16.6322	17.9879
16.6322	1	14	1	1	1	0	17.0998	17.9879
17.0998	1	15	1	1	1	1	17.5842	17.9879
17.5842	1	16	1	1	1	2	19.5915	17.9879
17.9879	0	6	1	0	1	0	19.5915	19.1843
19.1843	0	7	0	0	1	0	19.5915	22.5107

The first column represents the simulation clock. In the second column the type of event has been specified. “0” indicates “departure”, whereas “1” indicates “arrival”. Each customer is numbered and these numbers can be seen in column three. The following three columns represent the status of the servers (1=busy, 0=idle). Column seven shows the number of customers lost. Last two columns signify the next arrival and departure times. For example, the second arrival into the system occurs at time 1.86. This customer is served by the first server. There have not been any lost customers yet. The next arrival, the third arrival, will be at time 1.9217, and the customer who is being served at that time will depart from the system at time 11.0629.

Exponential, Weibull, and Gamma distributions were considered as the interarrival time distribution. The computations were carried out for Weibull distribution with different parameters. The system simulation was replicated different times for each interarrival time distribution, and the loss probability was computed as the relative frequency of the loss probability found in each run for each case. It is assumed 1000 customers for the each simulation run. The loss probabilities when the interarrival time comes from Exponential, Gamma and Weibull were compared with those for the case in which the interarrival time is deterministic.

The following three tables provide the loss probability for different interarrival time distributions with three different means.

In the first case, mean interarrival time was assumed to be 2 minutes. Three different interarrival time distributions such as exponential, gamma, and weibull with the same mean were compared to the case in which interarrival times are deterministic with a mean of 2 minutes. The results are shown in Table 2. Simulation was performed 250, 500, 1000, and 5000 times and the loss probability was estimated for each run and it was concluded that this probability does not chance according to the replication number. Besides, the standard error of estimates and 95% confidence intervals can be seen in the last column of the Table 2. As we can see, the loss probability is minimized when interarrival time is deterministic with a mean of 2 minutes.

The probability density function of Weibull distribution is given as follows:

$$f(x) = abx^{b-1}e^{-ax^b}, \quad x \geq 0$$

Weibull distribution was considered with two different shape parameter; *b*=3 and *b*=4. Scale parameter *a* was determined so that the mean was 2 minutes.

The probability density function of Gamma distribution is given as follows:

$$f(x) = x^{k-1}\frac{e^{-x/\theta}}{\theta^k \Gamma(k)}, \quad x > 0$$

The shape parameter *k* and the scale parameter θ were determined so that mean was 2 minutes.
The same computations were performed for the same interarrival time distributions with a mean of 4 minutes and the results are shown in Table 3. The mean server times for servers are fixed, as before. The simulation results show that the loss probability is minimized when interarrival time is deterministic with a mean of 4 minutes.

Table 2. The loss probabilities when the mean interarrival time is 2 minutes.

Interarrival Time Distribution	Mean interarrival Time (minutes)	Replication number	Loss Probability	SE of estimates and 95% CI
Exponential (λ=1/2)	2	250	0.1874	0.0097 (0.1684 ; 0.2063)
		500	0.1892	0.0048 (0.1798 ; 0.1987)
		1000	0.1867	0.0024 (0.1820 ; 0.1915)
		2000	0.1879	0.0012 (0.1855 ; 0.1903)
		5000	0.1889	0.0005 (0.1880 ; 0.1899)
Gamma (k=2,θ=1)	2	250	0.1233	0.0019 (0.1196 ; 0.1270)
		500	0.1221	0.0012 (0.1198 ; 0.1244)
		1000	0.1228	0.0008 (0.1213 ; 0.1244)
		2000	0.1235	0.0005 (0.1224 ; 0.1245)
		5000	0.1231	0.0003 (0.1224 ; 0.1238)
Weibull (a=0.09;b=3)	2	250	0.0811	0.001 (0.0791 ; 0.0830)
		500	0.0795	0.0007 (0.0781 ; 0.0809)
		1000	0.0794	0.0005 (0.0783 ; 0.0804)
		2000	0.0800	0.0004 (0.0793 ; 0.0807)
		5000	0.0801	0.0002 (0.0796 ; 0.0805)
Weibull (a=0.042;b=4)	2	250	0.0717	0.0043 (0.0633 ; 0.0800)
		500	0.0720	0.021 (0.0678 ; 0.0762)
		1000	0.0724	0.0011 (0.0703 ; 0.0745)
		2000	0.0721	0.0005 (0.0710 ; 0.0731)
		5000	0.0716	0.0002 (0.0712 ; 0.0721)
Deterministic	2	250	0.0616	0.0009 (0.0599 ; 0.0634)
		500	0.0619	0.0006 (0.0607 ; 0.0631)
		1000	0.0624	0.0004 (0.0615 ; 0.0632)
		2000	0.0626	0.0003 (0.0620 ; 0.0632)
		5000	0.0624	0.0002 (0.0620 ; 0.0628)

Table 3. The loss probabilities when the mean interarrival time is 4 minutes.

Interarrival Time Distribution	Mean interarrival Time (minutes)	Replication number	Loss Probability	SE of estimates and %95 CI
Exponential (λ=1/4)	4	250	0.0408	0.0034 (0.0342 ; 0.0474)
		500	0.0425	0.0017 (0.0392 ; 0.0458)
		1000	0.0418	0.0008 (0.0402 ; 0.0435)
		2000	0.0418	0.0004 (0.0410 ; 0.0426)
		5000	0.0413	0.0002 (0.0410 ; 0.0417)
Gamma (k=2,θ=2)	4	250	0.0159	0.004 (0.0081 ; 0.0237)
		500	0.0158	0.0019 (0.0121 ; 0.0194)
		1000	0.0162	0.0008 (0.0147 ; 0.0177)
		2000	0.0160	0.0001 (0.0158 ; 0.0163)
		5000	0.0161	0.0001 (0.0159 ; 0.0162)
Weibull (a=0.011;b=3)	4	250	0.0050	0.003 (-0.0009 ; 0.0109)
		500	0.0048	0.0015 (0.0018 ; 0.0078)
		1000	0.0047	0.0008 (0.0032 ; 0.0062)
		2000	0.0048	0.0003 (0.0041 ; 0.0054)
		5000	0.0047	0.0000 (0.0047 ; 0.0048)
Weibull (a=0.0026;b=4)	4	250	0.0035	0.0001 (0.0032 ; 0.0037)
		500	0.0032	0.0001 (0.0030 ; 0.0034)
		1000	0.0035	0.0001 (0.0034 ; 0.0037)
		2000	0.0033	0.000 (0.0033 ; 0.0034)
		5000	0.0034	0.0000 (0.0034 ; 0.0035)
Deterministic	4	250	0.0019	0.0023 (-0.0026 ; 0.0064)
		500	0.0020	0.0011 (-0.0001 ; 0.0041)
		1000	0.0020	0.0004 (0.0012 ; 0.0029)
		2000	0.0020	0.00001 (0.0019 ; 0.0021)
		5000	0.0019	0.00001 (0.00189 ; 0.00191)

Finally, the calculations for the same interarrival time distributions with a mean of 6 minutes were performed, and the results are shown in Table 4. The simulation results show that the loss probability is minimized when interarrival time is deterministic with a mean of 6 minutes.

Table 4. The loss probabilities when the mean interarrival time is 6 minutes.

Interarrival Time Distribution	Mean interarrival Time (minutes)	Replication number	Loss Probability	SE of estimates and %95 CI
Exponential (λ=1/6)	6	250	0.0152	0.0004 (0.0144 ; 0.0160)
		500	0.0153	0.0003 (0.0148 ; 0.0158)
		1000	0.0152	0.0002 (0.0148 ; 0.0156)
		2000	0.0154	0.0001 (0.0151 ; 0.0156)
		5000	0.0152	0.0001 (0.0150 ; 0.0154)
Gamma (k=2,θ=3)	6	250	0.0038	0.001 (0.0018 ; 0.0058)
		500	0.0035	0.0005 (0.0026 ; 0.0045)
		1000	0.0036	0.0002 (0.0032 ; 0.0040)
		2000	0.0038	0.0001 (0.0037 ; 0.0039)
		5000	0.0037	0.0000 (0.0036 ; 0.0037)
Gamma (k=3,θ=2)	6	250	0.0016	0.0004 (0.0008 ; 0.0024)
		500	0.0016	0.0002 (0.0012 ; 0.0020)
		1000	0.0017	0.0001 (0.0015 ; 0.0019)
		2000	0.0017	0.0001 (0.0015 ; 0.0018)
		5000	0.0017	0.0000 (0.0017 ; 0.0017)
Weibull (a=0.0033;b=3)	6	250	0.0005	0.0006 (-0.0007 ; 0.0018)
		500	0.0005	0.0003 (-0.0001 ; 0.0011)
		1000	0.0005	0.0001 (0.0003 ; 0.0008)
		2000	0.0005	0.0001 (0.0004 ; 0.0007)
		5000	0.0005	0.0000 (0.0005 ; 0.0006)
Weibull (a=0.00052;b=4)	6	250	0.0003	0.0002 (-0.0002 ; 0.0007)
		500	0.0003	0.0001 (0.0000 ; 0.0005)
		1000	0.0003	0.0001 (0.0002 ; 0.0004)
		2000	0.0003	0.0000 (0.0002 ; 0.0003)
		5000	0.0003	0.0000 (0.0003 ; 0.0003)
Deterministic	6	250	0.0001	0.0003 (-0.0004 ; 0.0006)
		500	0.0001	0.0001 (-0.0002 ; 0.0003)
		1000	0.0001	0.0001 (-0.0000 ; 0.0002)
		2000	0.0001	0.0000 (0.0001 ; 0.0001)
		5000	0.0001	0.0000 (0.0001 ; 0.0001)

4. CONCLUSIONS AND SUGGESTIONS

By using semi-Markov process representing the $GI/\vec{M}/3/0$ queuing model, the basic characteristics of the system were computed. By analyzing the stream of overflow, the LS transforms of interover flow times were obtained. With these transforms, the means of the aforementioned times were computed. At the instants of the customers' arrival, a formula of loss probability was obtained. As this formula is defined with a determinant consisting of a one-step probability, it can be calculated easily. Assuming that the mean of the interarrival times distribution is fixed, it is indicated that the loss probability is minimized by deterministic interarrival time distribution.

The result of the theorem was supported by a simulation study in which there are three heterogeneous servers. Having shown that the mean service times are statistically different at a %1 significance level, the computations were conducted. The loss probability was computed for three statistical distributions of interarrival times and compared with the loss probability for deterministic interarrival time. The loss probability under deterministic interarrival time distribution is minimized in each case.

The $GI/\vec{M}/n/0$ queuing model can be analyzed by a similar method used in this paper. It is expected that the loss probability is minimized when the interarrival time distribution is deterministic for the $GI/\vec{M}/n/0$ queuing model.

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