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Electromagnetic energy conservation with complex octonions

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Abstract

Octonions are the eight dimensional hypercomplex numbers that form a noncommutative and nonassociative division algebra. In this study, a general framework for the real, complex octonions and their algebra are provided by using the Cayley-Dickson multiplication rule between the octonionic basis elements. Maxwell's equations without sources are shown in Gauss units in dimensionless form. The local energy conservation equation, which has been previously defined in a complexified quaternionic form, is similarly rearranged for isotropic media by using the complex octonions. As a result, the terms of density and flow of electromagnetic energy are attained.

Key Words: Octonion, Maxwell's equations, electromagnetic energy, Poynting vector

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1. Introduction

Quaternions and octonions are useful tool for the representations and generalizations of quantities in the high-dimensional physical theory. These algebraic structures are used in areas such as quantum physics, classical electrodynamics, the representations of robotic systems' kinematics, acoustics, wave and group theory, supersymmetric quantum mechanics etc. While quaternions have a four dimensional noncommutative but associative algebraic structure, the octonions possess eight components and both noncommutative and nonassociative algebraic properties. According to the case of physical structures, quaternions and octonions can be used in the real, complex, dual, split, hyperbolic forms with the different dimensions and algebraic properties. In literature, there are a lot of studies about on the classical electromagnetism with different algebras [1–16].

After defining electromagnetism and energy conservation with complexified quaternions in the electromagnetic field by Tanışlı [5], a study of the classical electromagnetism's energy will be alternatively described by the complex octonions in sixteen dimensions. As in previous studies, the equations are obtained for isotropic media.

Organization of this paper is as follows: Section 2 introduces the octonion algebra and its properties. Maxwell's equations without sources, the complex octonionic field equation and differential operator are presented in section 3. Hence, the detailed equations of electromagnetic energy conservation, electromagnetic energy flow and density are suggested via the octonionic representations. The results, conclusions and fundamental features of this study are drawn and emphasized in the last section.

2. Preliminaries

Octonions are from the family of hypercomplex numbers and have eight dimensions. They were found in 1843 by John. T. Graves, and two years later independently by Arthur Cayley who formally published his finding. They are therefore at times also called "Cayley numbers". They are a unique division algebra in that their multiplication is alternative but generally noncommutative and nonassociative [15–19]. Let A be an octonion, expressed as,

$$A=\sum_{n=0}^{7} a_n \boldsymbol{e}_n = a_0\boldsymbol{e}_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4 + a_5\boldsymbol{e}_5 + a_6\boldsymbol{e}_6 + a_7\boldsymbol{e}_7, \tag{1}$$

where terms a_n are real number coefficients of the octonion and the $\boldsymbol{e}_n$'s are its basis elements. For two octonions such as A and B, the summation and substraction processes are given as

$$\begin{aligned} A \pm B &= \sum_{n=0}^{7} (a_n \pm b_n)\, \boldsymbol{e}_n \\ &= (a_0\boldsymbol{e}_0 + a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4 + a_5\boldsymbol{e}_5 + a_6\boldsymbol{e}_6 + a_7\boldsymbol{e}_7) \\ &\quad \pm (b_0\boldsymbol{e}_0 + b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 + b_4\boldsymbol{e}_4 + b_5\boldsymbol{e}_5 + b_6\boldsymbol{e}_6 + b_7\boldsymbol{e}_7)\,. \end{aligned} \tag{2}$$

The octonion A has scalar and vectorial parts as well. For defined octonion in equation (1), the scalar and vectorial parts can be given, respectively, as

$$\mathrm{S}_A = a_0\boldsymbol{e}_0; \tag{3}$$

$$\mathrm{V}_A = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4 + a_5\boldsymbol{e}_5 + a_6\boldsymbol{e}_6 + a_7\boldsymbol{e}_7. \tag{4}$$

Therefore, the octonion A can be written briefly as

$$A = \mathrm{S}_A + \mathrm{V}_A = a_0\boldsymbol{e}_0 + \vec{\boldsymbol{A}}. \tag{5}$$

Many multiplication rules can be found in literature that all represent an octonion algebra. In this study, we chose a multiplication rule obtained from Cayley-Dickson construction with the following properties:

$$\begin{aligned} &-\boldsymbol{e}_4\boldsymbol{e}_i = \boldsymbol{e}_i\boldsymbol{e}_4 = \hat{\boldsymbol{e}}_i,\ \ \boldsymbol{e}_4\hat{\boldsymbol{e}}_i = -\hat{\boldsymbol{e}}_i\boldsymbol{e}_4 = \boldsymbol{e}_i,\ \ \boldsymbol{e}_4\boldsymbol{e}_4 = -\boldsymbol{e}_0, \\ &\boldsymbol{e}_i\boldsymbol{e}_j = -\delta_{ij}\boldsymbol{e}_0 + \varepsilon_{ijk}\boldsymbol{e}_k,\ \ \hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = -\delta_{ij}\boldsymbol{e}_0 - \varepsilon_{ijk}\boldsymbol{e}_k,\ \ i,j,k \in (1,2,3)\,, \\ &-\hat{\boldsymbol{e}}_j\boldsymbol{e}_i = \boldsymbol{e}_i\hat{\boldsymbol{e}}_j = -\delta_{ij}\boldsymbol{e}_4 - \varepsilon_{ijk}\hat{\boldsymbol{e}}_k. \end{aligned} \tag{6}$$

Here, $\hat{\boldsymbol{e}}_k \equiv \boldsymbol{e}_{4+k}$, $k \in (1,2,3)$ and $\boldsymbol{e}_0 = 1$ [16]. These rules can also be introduced in tabular form, and is presented as such in Table.

Table. Cayley-Dickson multiplication rules for octonions.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

By using these rules, the product of octonions, A and B is expressed as

$$AB = a_0 b_0 + a_0 \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}} b_0 - \vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} + \vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}. \tag{7}$$

Octonions also have an octonionic conjugate which is denoted by changing the signs of the vectorial parts:

$$\bar{A} = \mathrm{S}_A - \mathrm{V}_A = a_0 \boldsymbol{e}_0 - \vec{\boldsymbol{A}}. \tag{8}$$

According to the octonionic conjugate process for octonions A and B, one can expresses the following rules:

$$\left(\bar{\bar{A}}\right) = A, \left(\overline{A+B}\right) = \bar{A} + \bar{B}, \left(\overline{AB}\right) = \bar{B}\bar{A}. \tag{9}$$

If there is no scalar parts in equation (7), the scalar and vectorial products of the octonions A and B are given the manner, respectively, [15, 16, 19]

$$\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = -\frac{1}{2}\left[AB + \left(\overline{AB}\right)\right], \tag{10}$$

$$\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = \frac{1}{2}\left[AB - \left(\overline{AB}\right)\right]. \tag{11}$$

The norm of the octonion A is obtained by multiplying the octonion and its octonionic conjugate; the result of this norm is a real number:

$$\mathrm{N}(A) = A\bar{A} = \bar{A}A = \sum_{n=0}^{7} a_n^2. \tag{12}$$

For two octonions, the norm is multiplicative:

$$\mathrm{N}(AB) = \mathrm{N}(A)\,\mathrm{N}(B). \tag{13}$$

Nonzero octonions also have a multiplicative inverse. The inverse of the octonion, A, is denoted by A^{-1} and this term can be obtained by the norm and the conjugate of the octonion:

$$A^{-1} = \frac{\bar{A}}{\mathrm{N}(A)}. \tag{14}$$

A complex octonion $\mathbf{X}$ can be understood as a combination of two octonions A and A' with a new unit i:

$$\mathbf{X} = A + \mathrm{i}A' \tag{15}$$

$$\begin{aligned}\mathbf{X} &= (a_0\boldsymbol{e}_0+a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3+a_4\boldsymbol{e}_4+a_5\boldsymbol{e}_5+a_6\boldsymbol{e}_6+a_7\boldsymbol{e}_7)\\ &\quad +\mathrm{i}\left(a_0'\boldsymbol{e}_0+a_1'\boldsymbol{e}_1+a_2'\boldsymbol{e}_2+a_3'\boldsymbol{e}_3+a_4'\boldsymbol{e}_4+a_5'\boldsymbol{e}_5+a_6'\boldsymbol{e}_6+a_7'\boldsymbol{e}_7\right),\end{aligned} \tag{16}$$

$$\begin{aligned}\mathbf{X} &= \sum_{n=0}^{7}\left(a_n+\mathrm{i}a_n'\right)\boldsymbol{e}_n=\left(a_0+\mathrm{i}a_0'\right)\boldsymbol{e}_0+\left(a_1+\mathrm{i}a_1'\right)\boldsymbol{e}_1+\left(a_2+\mathrm{i}a_2'\right)\boldsymbol{e}_2+\left(a_3+\mathrm{i}a_3'\right)\boldsymbol{e}_3\\ &\quad +\left(a_4+\mathrm{i}a_4'\right)\boldsymbol{e}_4+\left(a_5+\mathrm{i}a_5'\right)\boldsymbol{e}_5+\left(a_6+\mathrm{i}a_6'\right)\boldsymbol{e}_6+\left(a_7+\mathrm{i}a_7'\right)\boldsymbol{e}_7,\end{aligned} \tag{17}$$

$$\mathbf{X}=\sum_{n=0}^{7}\mathbf{x}_n\boldsymbol{e}_n=\mathbf{x}_0\boldsymbol{e}_0+\mathbf{x}_1\boldsymbol{e}_1+\mathbf{x}_2\boldsymbol{e}_2+\mathbf{x}_3\boldsymbol{e}_3+\mathbf{x}_4\boldsymbol{e}_4+\mathbf{x}_5\boldsymbol{e}_5+\mathbf{x}_6\boldsymbol{e}_6+\mathbf{x}_7\boldsymbol{e}_7. \tag{18}$$

Here, the $\mathbf{x}_n$'s are complex numbers and i denotes the complex unit $\left(\mathrm{i}=\sqrt{-1}\right)$. While complex octonions have similar algebraic properties as the octonions, they differ by having 16 dimensions and an additional complex unit i. This means that there exists an additional complex conjugate of a complex octonion. Octonion conjugate $\bar{\mathbf{X}}$ and complex conjugate $\mathbf{X}^*$ are written as:

$$\begin{aligned}\bar{\mathbf{X}} &= \left(a_0+\mathrm{i}a_0'\right)\boldsymbol{e}_0-\left(a_1+\mathrm{i}a_1'\right)\boldsymbol{e}_1-\left(a_2+\mathrm{i}a_2'\right)\boldsymbol{e}_2-\left(a_3+\mathrm{i}a_3'\right)\boldsymbol{e}_3\\ &\quad -\left(a_4+\mathrm{i}a_4'\right)\boldsymbol{e}_4-\left(a_5+\mathrm{i}a_5'\right)\boldsymbol{e}_5-\left(a_6+\mathrm{i}a_6'\right)\boldsymbol{e}_6-\left(a_7+\mathrm{i}a_7'\right)\boldsymbol{e}_7,\end{aligned} \tag{19}$$

$$\begin{aligned}\mathbf{X}^* &= \left(a_0-\mathrm{i}a_0'\right)\boldsymbol{e}_0+\left(a_1-\mathrm{i}a_1'\right)\boldsymbol{e}_1+\left(a_2-\mathrm{i}a_2'\right)\boldsymbol{e}_2+\left(a_3-\mathrm{i}a_3'\right)\boldsymbol{e}_3\\ &\quad +\left(a_4'-\mathrm{i}a_4'\right)\boldsymbol{e}_4+\left(a_5-\mathrm{i}a_5'\right)\boldsymbol{e}_5+\left(a_6-\mathrm{i}a_6'\right)\boldsymbol{e}_6+\left(a_7-\mathrm{i}a_7'\right)\boldsymbol{e}_7.\end{aligned} \tag{20}$$

In Cayley-Dickson notation, the complex octonionic differential operator and its octonionic conjugate are defined in literature [15, 16, 19]:

$$\mathbf{D}_t=\frac{\mathrm{i}}{c}\frac{\partial}{\partial t}\boldsymbol{e}_0+\frac{\partial}{\partial x}\boldsymbol{e}_5+\frac{\partial}{\partial y}\boldsymbol{e}_6+\frac{\partial}{\partial z}\boldsymbol{e}_7, \tag{21}$$

$$\overline{\mathbf{D}}_t=\frac{\mathrm{i}}{c}\frac{\partial}{\partial t}\boldsymbol{e}_0-\frac{\partial}{\partial x}\boldsymbol{e}_5-\frac{\partial}{\partial y}\boldsymbol{e}_6-\frac{\partial}{\partial z}\boldsymbol{e}_7. \tag{22}$$

The multiplication of equations (21) and (22) is commutative and the result is equal to

$$\mathbf{D}_t\overline{\mathbf{D}_t}=\overline{\mathbf{D}_t}\mathbf{D}_t=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \tag{23}$$

where the symbols Δ and c present the Laplacian operator and speed of light, respectively.

3. Maxwell's equations and octonionic electromagnetic energy conservation

Maxwell's equations are central for the description of classical electromagnetism and optics. In Gauss unit system, four Maxwell's equations in vectorial form are [20]

$$\begin{aligned}&\vec{\nabla}\cdot\vec{D}=4\pi\rho\\ &\vec{\nabla}\cdot\vec{B}=0\\ &\vec{\nabla}\times\vec{E}=-\tfrac{1}{c}\tfrac{\partial\vec{B}}{\partial t}\\ &\vec{\nabla}\times\vec{H}=\tfrac{4\pi}{c}\vec{J}+\tfrac{1}{c}\tfrac{\partial\vec{D}}{\partial t},\end{aligned} \tag{24}$$

where $\vec{D}$, $\vec{E}$, $\vec{B}$, $\vec{H}$, ρ, $\vec{J}$ and c are used for defining of the electrical flux density, electric field, magnetic flux density, magnetic field, electrical charge density, electrical current density and speed of light, respectively. For the isotropic media in electromagnetism, the constitutive equations as $\vec{D} = \varepsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$ are valid. Here, the terms ε_0 and μ_0 represent permittivity and permeability constants for free space. There is an usual assumption as $\varepsilon_0 = \mu_0 = c = 1$ for theoretical studies in physics, and then Maxwell's equations will be

$$\begin{aligned} &\vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ &\vec{\nabla} \cdot \vec{B} = 0 \\ &\vec{\nabla} \times \vec{E} = -\tfrac{\partial \vec{B}}{\partial t} \\ &\vec{\nabla} \times \vec{B} = 4\pi\vec{J} + \tfrac{\partial \vec{D}}{\partial t}. \end{aligned} \tag{25}$$

In equation (25), if the charge and current densities are equal to zero, Maxwell's equations in dimensionless form can be rewritten as the following equations:

$$\begin{aligned} &\vec{\nabla} \cdot \vec{E} = 0 \\ &\vec{\nabla} \cdot \vec{B} = 0 \\ &\vec{\nabla} \times \vec{E} = -\tfrac{\partial \vec{B}}{\partial t} \\ &\vec{\nabla} \times \vec{B} = \tfrac{\partial \vec{E}}{\partial t}. \end{aligned} \tag{26}$$

From the third and fourth Maxwell's equations in equation (26), the expression for the electromagnetic energy density (termed the Poynting Theorem) may be derived from:

$$-\vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right) + \vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) + \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = 0. \tag{27}$$

Using the vectorial identities as $\vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B}\right)$, this equation may then be described as the conservation law for electromagnetic energy as

$$\frac{\partial \mathrm{u}}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0 \quad , \tag{28}$$

where

$$\vec{S} = \vec{E} \times \vec{B} \tag{29}$$

is termed the Poynting vector. The rate of change of the energy density $\frac{\partial \mathrm{u}}{\partial t}$ is then defined as

$$\frac{\partial \mathrm{u}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \left(E^2 + B^2\right) = \frac{1}{2} \frac{\partial}{\partial t} \left(\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}\right). \tag{30}$$

At this stage, a field $\mathbf{F}$ that describes the electric and magnetic fields in isotropic media can be defined in complex octonion form:

$$\mathbf{F} = E + \mathrm{i}B = \left(E_x \boldsymbol{e}_5 + E_y \boldsymbol{e}_6 + E_z \boldsymbol{e}_7\right) + \mathrm{i}\left(B_x \boldsymbol{e}_1 + B_y \boldsymbol{e}_2 + B_z \boldsymbol{e}_3\right). \tag{31}$$

Maxwell's equations, which are denoted in equation (26), can then be written directly from the octonion product:

$$\mathbf{D}_t \mathbf{F} = 0. \tag{32}$$

As known from the complexified quaternionic Lagrange density [5], an equation can be suggested in the complex octonionic form as

$$\mathbf{F}^{*}\cdot\left(\mathbf{D}_{t}\mathbf{F}\right)=0. \tag{33}$$

Equation (33) can be considered as another way in field theories for supporting of studies in literature. Using the components of the field and differential operator, if equation (33) is clearly written, and then we present

$$\begin{aligned}\mathbf{F}^{*}\cdot\left(\mathbf{D}_{t}\mathbf{F}\right) &= \left[\left(E_{x}\boldsymbol{e}_{5}+E_{y}\boldsymbol{e}_{6}+E_{z}\boldsymbol{e}_{7}\right)-\mathrm{i}\left(B_{x}\boldsymbol{e}_{1}+B_{y}\boldsymbol{e}_{2}+B_{z}\boldsymbol{e}_{3}\right)\right]\cdot\\ &\cdot\left\{\left(\mathrm{i}\frac{\partial}{\partial t}\boldsymbol{e}_{0}+\nabla\right)\left[\left(E_{x}\boldsymbol{e}_{5}+E_{y}\boldsymbol{e}_{6}+E_{z}\boldsymbol{e}_{7}\right)+\mathrm{i}\left(B_{x}\boldsymbol{e}_{1}+B_{y}\boldsymbol{e}_{2}+B_{z}\boldsymbol{e}_{3}\right)\right]\right\}=0,\end{aligned} \tag{34}$$

where dot "$\cdot$" denotes the scalar product of complex octonion.

In above equations, we take into consideration the Cayley-Dickson method for the multiplications of the complex octonionic basis, and the octonionic $\mathbf{F}^{*}\left(\mathbf{D}_{t}\mathbf{F}\right)$ and $\overline{\mathbf{F}^{*}\left(\mathbf{D}_{t}\mathbf{F}\right)}$ terms can be written as:

$$\begin{aligned}
&\mathrm{i}\boldsymbol{e}_{0}\left[-B\cdot\frac{\partial B}{\partial t}-B_{x}\left(\nabla\times E\right)_{x}-B_{y}\left(\nabla\times E\right)_{y}-B_{z}\left(\nabla\times E\right)_{z}-E\cdot\frac{\partial E}{\partial t}+E_{x}\left(\nabla\times B\right)_{x}+E_{y}\left(\nabla\times B\right)_{y}+E_{z}\left(\nabla\times B\right)_{z}\right]\\
&+\mathrm{i}\boldsymbol{e}_{1}\left[B_{x}\left(\nabla\cdot E\right)+B_{y}\frac{\partial B_{z}}{\partial t}+B_{y}\left(\nabla\times E\right)_{z}-B_{z}\frac{\partial B_{y}}{\partial t}-B_{z}\left(\nabla\times E\right)_{y}-E_{x}\left(\nabla\cdot B\right)-E_{y}\frac{\partial E_{z}}{\partial t}+E_{y}\left(\nabla\times B\right)_{z}+E_{z}\frac{\partial E_{y}}{\partial t}-E_{z}\left(\nabla\times B\right)_{y}\right]\\
&+\mathrm{i}\boldsymbol{e}_{2}\left[-B_{x}\left(\nabla\times E\right)_{z}-B_{x}\frac{\partial B_{z}}{\partial t}+B_{y}\left(\nabla\cdot E\right)+B_{z}\frac{\partial B_{x}}{\partial t}+B_{z}\left(\nabla\times E\right)_{x}+E_{x}\frac{\partial E_{z}}{\partial t}-E_{x}\left(\nabla\times B\right)_{z}-E_{y}\left(\nabla\cdot B\right)-E_{z}\frac{\partial E_{x}}{\partial t}+E_{z}\left(\nabla\times B\right)_{x}\right]\\
&+\mathrm{i}\boldsymbol{e}_{3}\left[B_{x}\frac{\partial B_{y}}{\partial t}+B_{x}\left(\nabla\times E\right)_{y}-B_{y}\frac{\partial B_{x}}{\partial t}-B_{y}\left(\nabla\times E\right)_{x}+B_{z}\left(\nabla\cdot E\right)-E_{x}\frac{\partial E_{y}}{\partial t}+E_{x}\left(\nabla\times B\right)_{y}+E_{y}\frac{\partial E_{x}}{\partial t}-E_{y}\left(\nabla\times B\right)_{x}-E_{z}\left(\nabla\cdot B\right)\right]\\
&+\boldsymbol{e}_{4}\left[-B\cdot\frac{\partial E}{\partial t}+B_{x}\left(\nabla\times B\right)_{x}+B_{y}\left(\nabla\times B\right)_{y}+B_{z}\left(\nabla\times B\right)_{z}-E\cdot\frac{\partial B}{\partial t}-E_{x}\left(\nabla\times E\right)_{x}-E_{y}\left(\nabla\times E\right)_{y}-E_{z}\left(\nabla\times E\right)_{z}\right]\\
&+\boldsymbol{e}_{5}\left[B_{x}\left(\nabla\cdot B\right)-B_{y}\frac{\partial E_{z}}{\partial t}+B_{y}\left(\nabla\times B\right)_{z}+B_{z}\frac{\partial E_{y}}{\partial t}-B_{z}\left(\nabla\times B\right)_{y}-E_{x}\left(\nabla\cdot E\right)+E_{y}\frac{\partial B_{z}}{\partial t}+E_{y}\left(\nabla\times E\right)_{z}-E_{z}\frac{\partial B_{y}}{\partial t}-E_{z}\left(\nabla\times E\right)_{y}\right]\\
&+\boldsymbol{e}_{6}\left[B_{x}\frac{\partial E_{z}}{\partial t}-B_{x}\left(\nabla\times B\right)_{z}+B_{y}\left(\nabla\cdot B\right)-B_{z}\frac{\partial E_{x}}{\partial t}+B_{z}\left(\nabla\times B\right)_{x}-E_{x}\frac{\partial B_{z}}{\partial t}-E_{x}\left(\nabla\times E\right)_{z}-E_{y}\left(\nabla\cdot E\right)+E_{z}\frac{\partial B_{x}}{\partial t}+E_{z}\left(\nabla\times E\right)_{x}\right]\\
&+\boldsymbol{e}_{7}\left[-B_{x}\frac{\partial E_{y}}{\partial t}+B_{x}\left(\nabla\times B\right)_{y}+B_{y}\frac{\partial E_{x}}{\partial t}-B_{y}\left(\nabla\times B\right)_{x}+B_{z}\left(\nabla\cdot B\right)+E_{x}\frac{\partial B_{y}}{\partial t}+E_{x}\left(\nabla\times E\right)_{y}-E_{y}\frac{\partial B_{x}}{\partial t}-E_{y}\left(\nabla\times E\right)_{x}-E_{z}\left(\nabla\cdot E\right)\right]
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\mathrm{i}\boldsymbol{e}_{0}\left[-B\cdot\frac{\partial B}{\partial t}-B_{x}\left(\nabla\times E\right)_{x}-B_{y}\left(\nabla\times E\right)_{y}-B_{z}\left(\nabla\times E\right)_{z}-E\cdot\frac{\partial E}{\partial t}+E_{x}\left(\nabla\times B\right)_{x}+E_{y}\left(\nabla\times B\right)_{y}+E_{z}\left(\nabla\times B\right)_{z}\right]\\
&-\mathrm{i}\boldsymbol{e}_{1}\left[B_{x}\left(\nabla\cdot E\right)+B_{y}\frac{\partial B_{z}}{\partial t}+B_{y}\left(\nabla\times E\right)_{z}-B_{z}\frac{\partial B_{y}}{\partial t}-B_{z}\left(\nabla\times E\right)_{y}-E_{x}\left(\nabla\cdot B\right)-E_{y}\frac{\partial E_{z}}{\partial t}+E_{y}\left(\nabla\times B\right)_{z}+E_{z}\frac{\partial E_{y}}{\partial t}-E_{z}\left(\nabla\times B\right)_{y}\right]\\
&-\mathrm{i}\boldsymbol{e}_{2}\left[-B_{x}\left(\nabla\times E\right)_{z}-B_{x}\frac{\partial B_{z}}{\partial t}+B_{y}\left(\nabla\cdot E\right)+B_{z}\frac{\partial B_{x}}{\partial t}+B_{z}\left(\nabla\times E\right)_{x}+E_{x}\frac{\partial E_{z}}{\partial t}-E_{x}\left(\nabla\times B\right)_{z}-E_{y}\left(\nabla\cdot B\right)-E_{z}\frac{\partial E_{x}}{\partial t}+E_{z}\left(\nabla\times B\right)_{x}\right]\\
&-\mathrm{i}\boldsymbol{e}_{3}\left[B_{x}\frac{\partial B_{y}}{\partial t}+B_{x}\left(\nabla\times E\right)_{y}-B_{y}\frac{\partial B_{x}}{\partial t}-B_{y}\left(\nabla\times E\right)_{x}+B_{z}\left(\nabla\cdot E\right)-E_{x}\frac{\partial E_{y}}{\partial t}+E_{x}\left(\nabla\times B\right)_{y}+E_{y}\frac{\partial E_{x}}{\partial t}-E_{y}\left(\nabla\times B\right)_{x}-E_{z}\left(\nabla\cdot B\right)\right]\\
&-\boldsymbol{e}_{4}\left[-B\cdot\frac{\partial E}{\partial t}+B_{x}\left(\nabla\times B\right)_{x}+B_{y}\left(\nabla\times B\right)_{y}+B_{z}\left(\nabla\times B\right)_{z}-E\cdot\frac{\partial B}{\partial t}-E_{x}\left(\nabla\times E\right)_{x}-E_{y}\left(\nabla\times E\right)_{y}-E_{z}\left(\nabla\times E\right)_{z}\right]\\
&-\boldsymbol{e}_{5}\left[B_{x}\left(\nabla\cdot B\right)-B_{y}\frac{\partial E_{z}}{\partial t}+B_{y}\left(\nabla\times B\right)_{z}+B_{z}\frac{\partial E_{y}}{\partial t}-B_{z}\left(\nabla\times B\right)_{y}-E_{x}\left(\nabla\cdot E\right)+E_{y}\frac{\partial B_{z}}{\partial t}+E_{y}\left(\nabla\times E\right)_{z}-E_{z}\frac{\partial B_{y}}{\partial t}-E_{z}\left(\nabla\times E\right)_{y}\right]\\
&-\boldsymbol{e}_{6}\left[B_{x}\frac{\partial E_{z}}{\partial t}-B_{x}\left(\nabla\times B\right)_{z}+B_{y}\left(\nabla\cdot B\right)-B_{z}\frac{\partial E_{x}}{\partial t}+B_{z}\left(\nabla\times B\right)_{x}-E_{x}\frac{\partial B_{z}}{\partial t}-E_{x}\left(\nabla\times E\right)_{z}-E_{y}\left(\nabla\cdot E\right)+E_{z}\frac{\partial B_{x}}{\partial t}+E_{z}\left(\nabla\times E\right)_{x}\right]\\
&-\boldsymbol{e}_{7}\left[-B_{x}\frac{\partial E_{y}}{\partial t}+B_{x}\left(\nabla\times B\right)_{y}+B_{y}\frac{\partial E_{x}}{\partial t}-B_{y}\left(\nabla\times B\right)_{x}+B_{z}\left(\nabla\cdot B\right)+E_{x}\frac{\partial B_{y}}{\partial t}+E_{x}\left(\nabla\times E\right)_{y}-E_{y}\frac{\partial B_{x}}{\partial t}-E_{y}\left(\nabla\times E\right)_{x}-E_{z}\left(\nabla\cdot E\right)\right].
\end{aligned} \tag{36}$$

Here, adding equation (35) and equation (36), and multiplying by $\left(-\frac{1}{2}\right)$, equation (34) is equal to

$$\begin{aligned}\mathbf{F}^{*}\cdot(\mathbf{D}_{t}\mathbf{F}) &= \mathrm{i}\boldsymbol{e}_0\left(E_x\frac{\partial E_x}{\partial t}+E_y\frac{\partial E_y}{\partial t}+E_z\frac{\partial E_z}{\partial t}\right)-\mathrm{i}\boldsymbol{e}_0\left(E_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)+E_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)\right.\\ &\quad\left.+E_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\right)+\mathrm{i}\boldsymbol{e}_0\left(B_x\frac{\partial B_x}{\partial t}+B_y\frac{\partial B_y}{\partial t}+B_z\frac{\partial B_z}{\partial t}\right)+\mathrm{i}\boldsymbol{e}_0\left(B_x\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\right.\\ &\quad\left.+B_y\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+B_z\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\right) \qquad (37)\\ &= -\frac{1}{2}\left[\mathbf{F}^{*}(\mathbf{D}_t\mathbf{F})+\overline{\mathbf{F}^{*}(\mathbf{D}_t\mathbf{F})}\right]=0,\end{aligned}$$

and in a more simple manner,

$$\mathbf{F}^{*}\cdot(\mathbf{D}_t\mathbf{F})=\mathrm{i}\boldsymbol{e}_0\left(E\cdot\frac{\partial E}{\partial t}+B\cdot\frac{\partial B}{\partial t}\right)+\mathrm{i}\boldsymbol{e}_0\left(B\cdot(\nabla\times E)-E\cdot(\nabla\times B)\right)=0. \qquad (38)$$

Equation (38) is also equal to the expression for conservation of the electromagnetic energy (termed the Poynting Theorem) presented in equations (27), (28) or (30). In other words, in equation (38), the terms $B\cdot(\nabla\times E)-E\cdot(\nabla\times B)$ and $E\cdot\frac{\partial E}{\partial t}+B\cdot\frac{\partial B}{\partial t}$ will be equal to $\nabla\cdot(E\times B)=\nabla\cdot S$ and $\frac{1}{2}\frac{\partial}{\partial t}(E\cdot E+B\cdot B)$, respectively. Here, S is the classical Poynting vector (electromagnetic energy flow) in complex octonionic form.

At this stage, the conservation of energy for classical electromagnetism is obtained as

$$\begin{aligned}\mathbf{F}^{*}\cdot(\mathbf{D}_t\mathbf{F}) &= \mathrm{i}\boldsymbol{e}_0\left(E\cdot\frac{\partial E}{\partial t}+B\cdot\frac{\partial B}{\partial t}\right)+\mathrm{i}\boldsymbol{e}_0\left(B\cdot(\nabla\times E)-E\cdot(\nabla\times B)\right)\\ &= \frac{1}{2}\frac{\partial}{\partial t}(E\cdot E+B\cdot B)+\nabla\cdot(E\times B)=\frac{\partial}{\partial t}(u)+\nabla\cdot S=0,\end{aligned} \qquad (39)$$

where u is also the octonionic electromagnetic energy density. The conservation equation for the electromagnetic energy with a non-associative algebra has been attained once again.

4. Conclusions

The known procedure for deriving local field conservation laws is to apply Noether's theorem, making use of the translational and rotational invariances of a second order Lagrangian. In this study, we have reformulated the complex octonionic field equation for classical electromagnetism and derived the local conservation equation for energy, thus eliminating the use of translational invariance. The classical Poynting Theorem is recovered from the complex octonionic field equation. The result obtained is consistent with known properties of classical electromagnetism, and advertises the usefulness of complex octonion algebra for this and similar physics problems. For example, the procedure used here to derive the local energy conservation equation for classical electromagnetism may be applied to derive the local linear momentum and local angular momentum conservation equations or the local energy conservation equation in bi-isotropic media with the Drude-Born-Fedorov equations for classical electromagnetism.

To obtain the conservation equations for the electromagnetism in different multiplication rules for the complex octonions' basis require of choosing the different field and differential operator in different basis.

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