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Almost contact metric structures induced by G_2 structures

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Abstract: We study almost contact metric structures induced by 2-fold vector cross products on manifolds with G_2 structures. We get some results on possible classes of almost contact metric structures. Finally, we give examples.

Key words: G_2 structure, almost contact metric structure

1. Introduction

A recent research area in geometry is the relation between manifolds with structure group G_2 and almost contact metric manifolds. A manifold with G_2 structure has a particular 3-form globally defined on its tangent bundle. Such manifolds were classified into sixteen classes by Fernández and Gray in [9] according to the properties of the covariant derivative of the 3-form.

On an almost contact metric manifold, there exists a global 2-form and the properties of the covariant derivative of this 2-form yield 2^{12} classes of almost contact metric manifolds [3, 7].

Recently Matzeu and Munteanu constructed almost contact metric structures induced by the 2-fold vector cross product on some classes of manifolds with G_2 structures admitting a globally defined nonzero vector field such as parallelizable 7-dimensional manifolds and orientable hypersurfaces of $\mathbb{R}^8$ [10]. Arikan et al. proved the existence of almost contact metric structures on manifolds with G_2 structures [4]. Todd studied almost contact metric structures on manifolds with parallel G_2 structures [12].

Our aim is to study almost contact metric structures on manifolds with arbitrary G_2 structures. We eliminate some classes that almost contact metric structures induced from a G_2 structure may belong to according to properties of the characteristic vector field of the almost contact metric structure. In particular, we also investigate the possible classes of almost contact metric structures on manifolds with nearly parallel G_2 structures. In addition, we give examples of almost contact metric structures on manifolds with G_2 structures induced by the 2-fold vector cross product.

2. Preliminaries

Consider $\mathbb{R}^7$ with the standard basis $\{e_1, ..., e_7\}$. The fundamental 3-form on $\mathbb{R}^7$ is defined as

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

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where $\{e^1, ..., e^7\}$ is the dual basis of the standard basis and $e^{ijk} = e^i \wedge e^j \wedge e^k$. Then the compact, simple, and simply connected 14-dimensional Lie group G_2 is

$$G_2 := \{f \in GL(7, \mathbb{R}) \mid f^*\varphi_0 = \varphi_0\}.$$

A manifold with G_2 structure is a 7-dimensional oriented manifold whose structure group reduces to the group G_2. In this case, there exists a global 3-form φ on M such that for all $p \in M$, $(T_pM, \varphi_p) \cong (\mathbb{R}^7, \varphi_0)$. This 3-form is called the fundamental 3-form or the G_2 structure on M and gives a Riemannian metric g, a volume form, and a 2-fold vector cross product P on M defined by

$$\varphi(x, y, z) = g(P(x, y), z) \tag{2.1}$$

for all vector fields x, y on M [6].

Manifolds (M, g) with G_2 structure φ were classified according to properties of the covariant derivative of the fundamental 3-form. The space

$$\mathcal{W} = \{\alpha \in (\mathbb{R}^7)^* \otimes \Lambda^3(\mathbb{R}^7)^* | \alpha(x, y \wedge z \wedge P(y, z)) = 0 \ \ \forall x, y, z \in \mathbb{R}^7\}$$

of tensors having the same symmetry properties as the covariant derivative of φ was given, and then this space was decomposed into four G_2-irreducible subspaces using the representation of the group G_2 on $\mathcal{W}$. Since

$$(\nabla\varphi)_p \in \mathcal{W}_p = \{\alpha \in T_p^*M \otimes \Lambda^3(T_p^*M) | \alpha(x, y \wedge z \wedge P(y, z)) = 0 \ \ \forall x, y, z \in T_pM\}$$

and there are 16 invariant subspaces of $\mathcal{W}_p$, each subspace corresponds to a different class of manifolds with G_2 structure. For example, the class $\mathcal{P}$, in which the covariant derivative of φ is zero, is the class of manifolds with parallel G_2 structure. A manifold in this class is sometimes called a G_2 manifold. $\mathcal{W}_1$ corresponds to the class of nearly parallel manifolds, which are manifolds with G_2 structure φ satisfying $d\varphi = k * \varphi$ for some constant k [9].

Let M^{2n+1} be a differentiable manifold of dimension $2n+1$. If there is a $(1,1)$ tensor field ϕ, a vector field ξ, and a 1-form η on M satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

then M is said to have an almost contact structure (ϕ, ξ, η). A manifold with an almost contact structure is called an almost contact manifold.

If in addition to an almost contact structure (ϕ, ξ, η), M also admits a Riemannian metric g such that

$$g(\phi(x), \phi(y)) = g(x, y) - \eta(x)\eta(y)$$

for all vector fields x, y, then M is an almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g). The Riemannian metric g is called a compatible metric. The 2-form Φ defined by

$$\Phi(x, y) = g(x, \phi(y))$$

for all vector fields x, y is called the fundamental 2-form of the almost contact metric manifold (M, ϕ, ξ, η, g).

In [7], a classification of almost contact metric manifolds was obtained via the study of the covariant derivative of the fundamental 2-form. Let (ξ, η, g) be an almost contact metric structure on $\mathbb{R}^{2n+1}$. A space

$$\begin{aligned}\mathcal{C} &= \{\alpha \in \otimes_3^0 \mathbb{R}^{2n+1} | \alpha(x,y,z) = -\alpha(x,z,y) = -\alpha(x,\phi y,\phi z) \\ &\qquad + \eta(y)\alpha(x,\xi,z) + \eta(z)\alpha(x,y,\xi)\}\end{aligned}$$

having the same symmetries as the covariant derivative of the fundamental 2-form was given. First this space was written as a direct sum of three subspaces,

$$\mathcal{D}_1 = \{\alpha \in \mathcal{C} | \alpha(\xi,x,y) = \alpha(x,\xi,y) = 0\}, \tag{2.2}$$

$$\mathcal{D}_2 = \{\alpha \in \mathcal{C} | \alpha(x,y,z) = \eta(x)\alpha(\xi,y,z) + \eta(y)\alpha(x,\xi,z) + \eta(z)\alpha(x,y,\xi)\}, \tag{2.3}$$

and

$$\mathcal{C}_{12} = \{\alpha \in \mathcal{C} | \alpha(x,y,z) = \eta(x)\eta(y)\alpha(\xi,\xi,z) + \eta(x)\eta(z)\alpha(\xi,y,\xi)\}, \tag{2.4}$$

and then $\mathcal{D}_1$, $\mathcal{D}_2$ were decomposed into $U(n)\times 1$ irreducible components $\mathcal{C}_1,\dots,\mathcal{C}_4$ and $\mathcal{C}_5,\dots,\mathcal{C}_{11}$, respectively. Thus, there are 2^{12} invariant subspaces given by all possible combinations of the twelve subspaces $\mathcal{C}_1,\dots,\mathcal{C}_{12}$, each corresponding to a class of almost contact metric manifolds. For example, the trivial class such that $\nabla\Phi = 0$ corresponds to the class of cosymplectic [5] (called co-Kähler by some authors) manifolds, $\mathcal{C}_1$, is the class of nearly-K-cosymplectic manifolds, etc.

In the classification of Chinea and Gonzales, it was shown that the space of quadratic invariants of $\mathcal{C}$ is generated by the following 18 elements:

$$\begin{array}{ll}
i_1(\alpha) = \sum\limits_{i,j,k} \alpha(e_i,e_j,e_k)^2 & i_2(\alpha) = \sum\limits_{i,j,k} \alpha(e_i,e_j,e_k)\alpha(e_j,e_i,e_k) \\
i_3(\alpha) = \sum\limits_{i,j,k} \alpha(e_i,e_j,e_k)\alpha(\phi e_i,\phi e_j,e_k) & i_4(\alpha) = \sum\limits_{i,j,k} \alpha(e_i,e_i,e_k)\alpha(e_j,e_j,e_k) \\
i_5(\alpha) = \sum\limits_{j,k} \alpha(\xi,e_j,e_k)^2 & i_6(\alpha) = \sum\limits_{i,k} \alpha(e_i,\xi,e_k)^2 \\
i_7(\alpha) = \sum\limits_{j,k} \alpha(\xi,e_j,e_k)\alpha(e_j,\xi,e_k) & i_8(\alpha) = \sum\limits_{i,j} \alpha(e_i,e_j,\xi)\alpha(e_j,e_i,\xi) \\
i_9(\alpha) = \sum\limits_{i,j} \alpha(e_i,e_j,\xi)\alpha(\phi e_i,\phi e_j,\xi) & i_{10}(\alpha) = \sum\limits_{i,j} \alpha(e_i,e_i,\xi)\alpha(e_j,e_j,\xi) \\
i_{11}(\alpha) = \sum\limits_{i,j} \alpha(e_i,e_j,\xi)\alpha(e_j,\phi e_i,\xi) & i_{12}(\alpha) = \sum\limits_{i,j} \alpha(e_i,e_j,\xi)\alpha(\phi e_j,\phi e_i,\xi) \\
i_{13}(\alpha) = \sum\limits_{j,k} \alpha(\xi,e_j,e_k)\alpha(\phi e_j,\xi,e_k) & i_{14}(\alpha) = \sum\limits_{i,j} \alpha(e_i,\phi e_i,\xi)\alpha(e_j,\phi e_j,\xi) \\
i_{15}(\alpha) = \sum\limits_{i,j} \alpha(e_i,\phi e_i,\xi)\alpha(e_j,e_j,\xi) & i_{16}(\alpha) = \sum\limits_{k} \alpha(\xi,\xi,e_k)^2 \\
i_{17}(\alpha) = \sum\limits_{i,k} \alpha(e_i,e_i,e_k)\alpha(\xi,\xi,e_k) & i_{18}(\alpha) = \sum\limits_{i,k} \alpha(e_i,e_i,\phi e_k)\alpha(\xi,\xi,e_k)
\end{array}$$

where $\{e_1, e_2, ..., e_{2n}, \xi\}$ is a local orthonormal basis. The following relations among quadratic invariants were also expressed for manifolds having dimensions ≥ 7, where $\alpha \in \mathcal{C}$ and $A = \{1,2,3,4,5,7,11,13,15,16,17,18\}$:

$\mathcal{C}_{\mathbf{1}}$:$i_1(\alpha) = -i_2(\alpha) = -i_3(\alpha) = ||\alpha||^2, \quad i_m(\alpha) = 0 \ (m \geq 4)$.

$\mathcal{C}_{\mathbf{2}}$:$i_1(\alpha) = 2i_2(\alpha) = -i_3(\alpha) = ||\alpha||^2, \quad i_m(\alpha) = 0 \ (m \geq 4)$.

$\mathcal{C}_{\mathbf{3}}$:$i_1(\alpha) = i_3(\alpha) = ||\alpha||^2, \quad i_2(\alpha) = i_m(\alpha) = 0 \ (m \geq 4)$.

$\mathcal{C}_{\mathbf{4}}$:$i_1(\alpha) = i_3(\alpha) = \frac{n}{(n-1)^2} i_4(\alpha) = \frac{n}{(n-1)^2} \sum\limits_{k}^{2n} c_{12}^2(\alpha)(e_k)$,

$i_2(\alpha) = i_m(\alpha) = 0\ (m > 4)$.
$\mathcal{C}_{\mathbf{5}}$:$i_6(\alpha) = -i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \frac{1}{2n} i_{14}(\alpha)$,
$i_{10}(\alpha) = i_m(\alpha) = 0\ \ (m \in A)$.
$\mathcal{C}_{\mathbf{6}}$:$i_6(\alpha) = i_8(\alpha) = i_9(\alpha) = i_{12}(\alpha) = \frac{1}{2n} i_{10}(\alpha)$,
$i_{14}(\alpha) = i_m(\alpha) = 0\ \ (m \in A)$.
$\mathcal{C}_{\mathbf{7}}$:$i_6(\alpha) = i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \frac{||\alpha||^2}{2}$,
$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0\ \ (m \in A)$.
$\mathcal{C}_{\mathbf{8}}$:$i_6(\alpha) = -i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \frac{||\alpha||^2}{2}$,
$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0\ \ (m \in A)$.
$\mathcal{C}_{\mathbf{9}}$:$i_6(\alpha) = i_8(\alpha) = -i_9(\alpha) = -i_{12}(\alpha) = \frac{||\alpha||^2}{2}$,
$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0\ \ (m \in A)$.
$\mathcal{C}_{\mathbf{10}}$:$i_6(\alpha) = -i_8(\alpha) = -i_9(\alpha) = i_{12}(\alpha) = \frac{||\alpha||^2}{2}$,
$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0\ \ (m \in A)$.
$\mathcal{C}_{\mathbf{11}}$:$i_5(\alpha) = ||\alpha||^2, i_m(\alpha) = 0\ \ (m \neq 5)$.
$\mathcal{C}_{\mathbf{12}}$:$i_{16}(\alpha) = ||\alpha||^2, i_m(\alpha) = 0\ \ (m \neq 16)$.

For details, refer to [7].

We give below the most studied classes of almost contact metric structures as the direct sum of spaces $\mathcal{C}_i$:

$$|\ C\ | = \text{the class of cosymplectic manifolds.}$$

$$\mathcal{C}_1 = \text{the class of nearly-K-cosymplectic manifolds.}$$

$$\mathcal{C}_2 \oplus \mathcal{C}_9 = \text{the class of almost cosymplectic manifolds.}$$

$$\mathcal{C}_5 = \text{the class of } \beta\text{-Kenmotsu manifolds.}$$

$$\mathcal{C}_6 = \text{the class of } \alpha\text{-Sasakian manifolds.}$$

$$\mathcal{C}_5 \oplus \mathcal{C}_6 = \text{the class of trans-Sasakian manifolds.}$$

$$\mathcal{C}_6 \oplus \mathcal{C}_7 = \text{the class of quasi-Sasakian manifolds.}$$

$$\mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 = \text{the class of semi-cosymplectic and normal manifolds.}$$

$$\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 = \text{the class of nearly trans-Sasakian manifolds.}$$

$$\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} = \text{the class of quasi-K-cosymplectic manifolds.}$$

$$\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 = \text{the class of normal manifolds.}$$

$$\mathcal{D}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} = \text{the class of almost-K-contact manifolds.}$$

$$\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11} = \text{the class of semi-cosymplectic manifolds.}$$

Note that the class $\mathcal{C}_{12}$ is not contained in the class of semi-cosymplectic manifolds [11].

Let (M, g) be a 7-dimensional Riemannian manifold with G_2 structure φ and the associated 2-fold vector cross product P, and let ξ be a nowhere zero vector field of unit length on M. Then for

$$\phi(x) := P(\xi, x) \qquad \eta(x) := g(\xi, x),$$

(ϕ, ξ, η, g) is an almost contact metric structure on M [4, 10]. Throughout this study, (ϕ, ξ, η, g) will denote the almost contact metric structure (a.c.m.s.) induced by the G_2 structure φ with the cross product P on M and Φ will denote the fundamental 2-form of the a.c.m.s. In addition, all vector fields are considered to be smooth.

3. Almost contact metric structures obtained from G_2 structures

Let M be a manifold with G_2 structure φ and ξ a nowhere zero unit vector field on M, and let (ϕ, ξ, η, g) be the a.c.m.s. with the fundamental form Φ induced by the G_2 structure φ.

If $\nabla\varphi = 0$, then it can be seen that $\nabla\Phi = 0$ if and only if $\nabla\xi = 0$ [2, 12].

If ξ is a Killing vector field on a manifold with any G_2 structure, then

$$\begin{aligned} d\eta(x, y) &= \frac{1}{2}\{(\nabla_x\eta)(y) - (\nabla_y\eta)(x)\} \\ &= \frac{1}{2}\{g(\nabla_x\xi, y) - g(\nabla_y\xi, x)\} \\ &= g(\nabla_x\xi, y), \end{aligned} \tag{3.1}$$

which implies

$$d\eta = 0 \Leftrightarrow \nabla\xi = 0.$$

Therefore, if the Killing vector field ξ is not parallel, then the a.c.m.s. cannot be nearly-K-cosymplectic ($\mathcal{C}_1$).

To deduce further results, we focus on the covariant derivative of the fundamental 2-form Φ, where the a.c.m.s. (ϕ, ξ, η, g) is obtained from a G_2 structure of any class and ξ is any nonzero vector field. Direct calculation gives

$$(\nabla_x\Phi)(y, z) = g(y, \nabla_x(P(\xi, z))) + g(\nabla_x z, P(\xi, y)). \tag{3.2}$$

We also compute some of $i_k(\nabla\Phi), (k = 1, ..., 18)$ to understand which class $\nabla\Phi$ may belong to.

Proposition 3.1 *Let φ be a G_2 structure on M of an arbitrary class and (ϕ, ξ, η, g) an a.c.m.s. obtained from φ. Then:*

a. $i_6(\nabla\Phi) = 0$ if and only if $\nabla_{e_i}\xi = 0$ for $i = 1, \cdots, 6$ (note that $\nabla_\xi\xi$ need not be zero),

b. $i_{16}(\nabla\Phi) = 0$ if and only if $\nabla_\xi\xi = 0$.

Proof By direct calculation, for any $i, k \in \{1, 2, ..., 6\}$,

$$\begin{aligned} (\nabla_{e_i}\Phi)(\xi, e_k) &= g(\xi, \nabla_{e_i}(P(\xi, e_k))) + g(\nabla_{e_i}e_k, P(\xi, \xi)) \\ &= g(\xi, \nabla_{e_i}(P(\xi, e_k))) \\ &= -g(\nabla_{e_i}\xi, P(\xi, e_k)) \end{aligned} \tag{3.3}$$

and thus we obtain

$$i_6(\nabla\Phi) = \sum_{i,k}((\nabla_{e_i}\Phi)(\xi, e_k))^2 = \sum_{i,k} g(\nabla_{e_i}\xi, P(\xi, e_k))^2. \tag{3.4}$$

Since $P(\xi, e_k) = e_l$ is a frame element other than ξ, we have

$$i_6(\nabla\Phi) = g(\nabla_{e_i}\xi, P(\xi, e_k))^2 = 0 \text{ iff } g(\nabla_{e_i}\xi, e_l) = 0 \text{ for } l = 1, \ldots, 6.$$

In addition, since $g(\xi, \xi) = 1$, we get $g(\nabla_{e_i}\xi, \xi) = 0$ for $i, \in \{1, 2, ..., 6\}$. Thus, $i_6(\nabla\Phi) = 0$ if and only if $\nabla_{e_i}\xi$ is zero $i, \in \{1, 2, ..., 6\}$.

Similarly,

$$\begin{aligned}(\nabla_\xi\Phi)(\xi, e_k) &= g(\xi, \nabla_\xi(P(\xi, e_k))) + g(\nabla_\xi e_k, P(\xi, \xi)) \\ &= -g(\nabla_\xi\xi, P(\xi, e_k)) \end{aligned} \tag{3.5}$$

for any $k \in \{1, 2, ..., 6\}$, and we get

$$i_{16}(\nabla\Phi) = \sum_k (\nabla_\xi\Phi)(\xi, e_k)^2 = \sum_k g(\nabla_\xi\xi, P(\xi, e_k))^2. \tag{3.6}$$

Note that $g(\nabla_\xi\xi, \xi) = 0$ since ξ is of unit length. As a result, $i_{16}(\nabla\Phi) = 0$ if and only if $\nabla_\xi\xi = 0$. □

Before giving results on possible classes of a.c.m.s. induced by G_2 structures, note that $\delta\eta = -div(\xi)$. To see this, consider the orthonormal basis $\{e_1, \cdots, e_6, \xi\}$. Then

$$\begin{aligned} div(\xi) &= \sum_{i=1}^{6} g(\nabla_{e_i}\xi, e_i) + g(\nabla_\xi\xi, \xi) \\ &= \sum_{i=1}^{6} g(\nabla_{e_i}\xi, e_i). \end{aligned} \tag{3.7}$$

On the other hand, since

$$\begin{aligned} (\nabla_{e_i}\eta)(e_i) &= e_i[\eta(e_i)] - \eta(\nabla_{e_i}e_i) \\ &= g(\nabla_{e_i}\xi, e_i) + g(\xi, \nabla_{e_i}e_i) - g(\xi, \nabla_{e_i}e_i) \\ &= g(\nabla_{e_i}\xi, e_i), \end{aligned} \tag{3.8}$$

we have

$$\delta\eta = -\sum_{i=1}^{6}(\nabla_{e_i}\eta)(e_i) = -\sum_{i=1}^{6} g(\nabla_{e_i}\xi, e_i) = -div(\xi). \tag{3.9}$$

Proposition 3.2 *Let (ϕ, η, ξ, g) be an almost contact metric structure induced by a G_2 structure φ. Then:*

- $i_{14}(\nabla\Phi) = 0$ *if and only if* $div(\xi) = 0$.
- $i_{15}(\nabla\Phi) = -div(\xi)g(\xi, v)$, *where* $v = \sum_{j=1}^{6} P(e_j, \nabla_{e_j}\xi)$.

Proof For any $i, j \in \{1, 2, ..., 6\}$ we have

$$\begin{aligned}(\nabla_{e_i}\Phi)(\phi e_i, \xi) &= g(P(\xi, e_i), \nabla_{e_i}(P(\xi, \xi))) + g(\nabla_{e_i}\xi, P(\xi, P(\xi, e_i))) \\ &= -g(\nabla_{e_i}\xi, e_i) \\ &= g(\xi, \nabla_{e_i} e_i). \qquad (3.10)\end{aligned}$$

On the other hand,

$$\sum_{i=1}^{6} \nabla_{e_i} e_i = -\sum_{i=1}^{6} div(e_i)e_i - div(\xi)\xi - \nabla_\xi \xi \qquad (3.11)$$

and thus

$$\begin{aligned}g(\xi, \sum_i \nabla_{e_i} e_i) &= -g(\xi, \sum_i div(e_i)e_i) - g(\xi, div(\xi)\xi) - g(\xi, \nabla_\xi \xi) \\ &= -div(\xi). \qquad (3.12)\end{aligned}$$

Then

$$\begin{aligned}i_{14}(\nabla\Phi) &= \sum_{i,j} (\nabla_{e_i}\Phi)(\phi e_i, \xi)(\nabla_{e_j}\Phi)(\phi e_j, \xi) \\ &= \Big(g(\xi, \sum_i \nabla_{e_i} e_i)\Big)\Big(g(\xi, \sum_j \nabla_{e_j} e_j)\Big) = (div(\xi))^2. \qquad (3.13)\end{aligned}$$

Therefore, $i_{14}(\nabla\Phi)$ is zero if and only if $div(\xi)$ is zero.

Similarly, from equations

$$(\nabla_{e_i}\Phi)(\phi e_i, \xi) = -g(\nabla_{e_i}\xi, e_i) \text{ and } (\nabla_{e_j}\Phi)(e_j, \xi) = g(\nabla_{e_j}\xi, P(\xi, e_j)),$$

we have

$$\begin{aligned}i_{15}(\nabla\Phi) &= \sum_{i,j} (\nabla_{e_i}\Phi)(\phi e_i, \xi)\ (\nabla_{e_j}\Phi)(e_j, \xi) \\ &= \sum_{i,j} g(\xi, \nabla_{e_i} e_i) g(\nabla_{e_j}\xi, P(\xi, e_j)) \\ &= \Big(g(\xi, \sum_i \nabla_{e_i} e_i)\Big)\Big(\sum_j g(\nabla_{e_j}\xi, P(\xi, e_j))\Big) \\ &= \Big(g(\xi, -div(\xi)\xi) - g(\xi, \sum_i div(e_i)e_i)\Big)\Big(\sum_j g(\xi, P(e_j, \nabla_{e_j}\xi))\Big) \\ &= -div(\xi).g(\xi, v). \qquad (3.14)\end{aligned}$$

Note that

$$g(\nabla_{e_j}\xi, P(\xi, e_j)) = \varphi(\xi, e_j, \nabla_{e_j}\xi) = \varphi(e_j, \nabla_{e_j}\xi, \xi) = g(P(e_j, \nabla_{e_j}\xi), \xi)$$

since φ is a 3-form. □

Now consider in particular an a.c.m.s. induced by a nearly parallel G_2 structure.

Proposition 3.3 *Let (ϕ, η, ξ, g) be an almost contact metric structure induced by a nearly parallel G_2 structure. Then:*

- $i_5(\nabla\Phi) = 0$ *if and only if* $\nabla_\xi \xi = 0$.
- *If* $\nabla_\xi \xi = 0$, *then* $i_{17}(\nabla\Phi) = i_{18}(\nabla\Phi) = 0$.

Proof Since φ is nearly parallel, for any $j, k \in \{1, 2, ..., 6\}$ we have

$$\begin{aligned}(\nabla_\xi \Phi)(e_j, e_k) &= g(e_j, \nabla_\xi(P(\xi, e_k))) + g(\nabla_\xi e_k, P(\xi, e_j)) \\ &= g(e_j, P(\nabla_\xi \xi, e_k)) + g(e_j, P(\xi, \nabla_\xi e_k)) + g(\nabla_\xi e_k, P(\xi, e_j)) \\ &= -g(\nabla_\xi \xi, P(e_j, e_k)). \end{aligned} \tag{3.15}$$

Thus,

$$i_5(\nabla\Phi) = \sum_{j,k} ((\nabla_\xi \Phi)(e_j, e_k))^2 = \sum_{j,k} (g(\nabla_\xi \xi, P(e_j, e_k)))^2, \tag{3.16}$$

which is zero if and only if $\nabla_\xi \xi$ is zero. Here $P(e_j, e_k)$ is also a frame element.

Similarly, for any $i, k \in \{1, 2, ..., 6\}$,

$$\begin{aligned}(\nabla_{e_i} \Phi)(e_i, \phi e_k) &= g(e_i, \nabla_{e_i}(P(\xi, P(\xi, e_k))) + g(\nabla_{e_i}(P(\xi, e_k)), P(\xi, e_i)) \\ &= g(e_i, \nabla_{e_i}(-e_k)) + g(\nabla_{e_i}(P(\xi, e_k)), P(\xi, e_i)) \\ &= g(\nabla_{e_i} e_i, e_k) + g(\nabla_{e_i}(P(\xi, e_k)), P(\xi, e_i)), \end{aligned} \tag{3.17}$$

$$\begin{aligned}(\nabla_\xi \Phi)(\xi, e_k) &= g(\xi, \nabla_\xi(P(\xi, e_k))) + g(\nabla_\xi e_k, P(\xi, \xi)) \\ &= g(\xi, P(\nabla_\xi \xi, e_k)) + g(\xi, P(\xi, \nabla_\xi e_k)) \\ &= -g(e_k, P(\nabla_\xi \xi, \xi)). \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned} i_{18}(\nabla\Phi) &= \sum_{i,k} ((\nabla_{e_i} \Phi)(e_i, \phi e_k))((\nabla_\xi \Phi)(\xi, e_k)) \\ &= -\sum_{i,k} \Big(g(\nabla_{e_i} e_i, e_k) + g(\nabla_{e_i}(P(\xi, e_k)), P(\xi, e_i)) \Big) \Big(g(e_k, P(\nabla_\xi \xi, \xi)) \Big) \\ &= -\sum_{i,k} \Big(g(\nabla_{e_i} e_i, e_k) g(e_k, P(\nabla_\xi \xi, \xi)) \Big) \\ &\quad - \sum_{i,k} \Big(g(\nabla_{e_i}(P(\xi, e_k)), P(\xi, e_i)) g(e_k, P(\nabla_\xi \xi, \xi)) \Big) \\ &= -\sum_{i,k} \Big(g(\nabla_{e_i} e_i, e_k) g(e_k, P(\nabla_\xi \xi, \xi)) \Big) \end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&+\sum_{i,k}\Big(g(\nabla_{e_i}e_i,e_k)g(e_k,P(\nabla_\xi\xi,\xi))\Big)\\
&-\sum_{i,k}\Big(g(P(\xi,e_k),P(e_i,\nabla_{e_i}\xi))g(e_k,P(\nabla_\xi\xi,\xi))\Big)\\
&=-\sum_i g\Big(P(\xi,(\sum_k g(P(\nabla_\xi\xi,\xi),e_k)e_k+g(P(\nabla_\xi\xi,\xi),\xi)\xi)),P(e_i,\nabla_{e_i}\xi)\Big)\\
&=-\sum_i g(P(\xi,P(\nabla_\xi\xi,\xi)),P(e_i,\nabla_{e_i}\xi))\\
&=-g(\nabla_\xi\xi,\sum_i(P(e_i,\nabla_{e_i}\xi))). \qquad (3.20)
\end{aligned}
$$

Thus, if $\nabla_\xi\xi$ is zero, so is $i_{18}(\nabla\Phi)$.

For i_{17} we compute

$$(\nabla_{e_i}\Phi)(e_i,e_k)=g(e_i,\nabla_{e_i}(P(\xi,e_k)))+g(\nabla_{e_i}e_k,P(\xi,e_i)) \qquad (3.21)$$

and

$$
\begin{aligned}
(\nabla_\xi\Phi)(\xi,e_k)&=g(\xi,\nabla_\xi(P(\xi,e_k)))+g(\nabla_\xi e_k,P(\xi,\xi))\\
&=-g(\nabla_\xi\xi,P(\xi,e_k))\\
&=g(e_k,P(\xi,\nabla_\xi\xi)) \qquad (3.22)
\end{aligned}
$$

for any $i,k\in\{1,2,...,6\}$ and we obtain

$$
\begin{aligned}
i_{17}(\nabla\Phi)&=\sum_{i,k}((\nabla_{e_i}\Phi)(e_i,e_k))((\nabla_\xi\Phi)(\xi,e_k))\\
&=\sum_{i,k}\Big(-g(\nabla_{e_i}e_i,P(\xi,e_k))-g(e_k,\nabla_{e_i}(P(\xi,e_i))\Big)\Big(g(e_k,P(\xi,\nabla_\xi\xi))\Big)\\
&=\sum_{i,k}g(e_k,P(\xi,\nabla_{e_i}e_i))g(e_k,P(\xi,\nabla_\xi\xi))-\sum_{i,k}g(e_k,P(\xi,\nabla_{e_i}e_i))g(e_k,P(\xi,\nabla_\xi\xi))\\
&+\sum_{i,k}g(e_k,P(e_i,\nabla_{e_i}\xi))g(e_k,P(\xi,\nabla_\xi\xi))\\
&=\sum_{i,k}g(e_k,P(e_i,\nabla_{e_i}\xi))g(e_k,P(\xi,\nabla_\xi\xi))\\
&=g(P(\xi,\nabla_\xi\xi),\sum_i P(e_i,\nabla_{e_i}\xi)). \qquad (3.23)
\end{aligned}
$$

Thus, if $\nabla_\xi\xi=0$, then $i_{17}(\nabla\Phi)=0$. □

Similarly, if $\nabla\xi$ is zero, then so is $i_{15}(\nabla\Phi)$; see Proposition 3.2.

Theorem 1 *Let M be a manifold with a G_2 structure φ and (ϕ,ξ,η,g) be an almost contact metric structure (a.c.m.s.) obtained from φ.*
(a) If $\nabla_\xi\xi\neq 0$, then $\nabla\Phi$ cannot be in classes $\mathcal{D}_2,\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_{11}$.

(b) If $div(\xi) \neq 0$, then the almost contact metric structure cannot belong to classes $\mathcal{D}_1$, $\mathcal{C}_i$ for $i = 1, 2, 3, 4, 6, 7, \cdots, 12$ and cannot be semi-cosymplectic ($\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11}$).

In the following proofs, we use the relations below given in [7] together with properties of i_m for each $\mathcal{C}_i$:
If $\alpha \in \mathcal{D}_1$, then $i_m(\alpha) = 0$ for $m \geq 5$.
If $\alpha \in \mathcal{D}_2$, then $i_m(\alpha) = 0$ for $m = 1, 2, 3, 4, 16, 17, 18$.

Proof (a) Let $\nabla_\xi \xi \neq 0$. Then by Proposition 3.1, we have $i_{16}(\nabla\Phi) \neq 0$. This implies $\nabla\Phi \notin \mathcal{D}_2$. In addition, $\nabla\Phi$ cannot belong to any of the classes C_i, $i = 1, \ldots, 11$.

(b)If $div(\xi) \neq 0$, then Proposition 3.2 yields that $i_{14}(\nabla\Phi) = (div(\xi))^2 \neq 0$. Hence, $\nabla\Phi$ cannot satisfy the defining relations of the classes

$$\mathcal{D}_1 = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4, \mathcal{C}_6, \cdots, \mathcal{C}_{12}.$$

Besides, the defining relation of semi-cosymplectic manifolds is

$$\delta\Phi = 0 \text{ and } \delta\eta = 0.$$

By equation (3.9), the a.c.m.s. is not semi-cosymplectic. □

Note that if $\nabla_\xi \xi \neq 0$, then since $\nabla\Phi \notin \mathcal{D}_2 = \mathcal{C}_5 \oplus \ldots \oplus \mathcal{C}_{11}$, the a.c.m.s. cannot be contained in any subclass of $\mathcal{D}_2$. In particular, the a.c.m.s. cannot be β-Kenmotsu, α-Sasakian, trans-Sasakian, or quasi-Sasakian.

If $div(\xi) \neq 0$, then we have $\nabla\Phi \notin \mathcal{D}_1 = \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_4$. In this case, the a.c.m.s. cannot be nearly-K-cosymplectic. Also, since the a.c.m.s. cannot be semi-cosymplectic, it cannot be almost-cosymplectic, β-Kenmotsu, α-Sasakian, trans-Sasakian, normal semi-cosymplectic, or quasi-K-cosymplectic.

Consider an a.c.m.s. induced by a nearly parallel G_2 structure. We deduce the following results.

Theorem 2 *Let (ϕ, ξ, η, g) be an a.c.m.s. obtained from a nearly parallel G_2 structure φ. If $\nabla_\xi \xi \neq 0$, then $\nabla\Phi$ cannot be in classes $\mathcal{D}_1, \mathcal{D}_2, \mathcal{C}_{12}$. ($\nabla\Phi$ may be contained by the classes $\mathcal{D}_1 \oplus \mathcal{D}_2, \mathcal{D}_1 \oplus \mathcal{C}_{12}, \mathcal{D}_2 \oplus \mathcal{C}_{12}, \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{C}_{12}$).*

Proof Let $\nabla_\xi \xi \neq 0$. By Proposition 3.3, $i_5(\nabla\Phi) \neq 0$. Thus, $\nabla\Phi$ cannot be in $\mathcal{D}_1$ and $\mathcal{C}_{12}$. Besides, by Proposition 3.1, we have $i_{16}(\nabla\Phi) \neq 0$, and then $\nabla\Phi$ cannot be in $\mathcal{D}_2$. □

In particular, the a.c.m.s. cannot belong to any subclasses of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Theorem 3 *Let (ϕ, ξ, η, g) be an a.c.m.s. obtained from a nearly parallel G_2 structure φ. Then $\nabla_\xi \xi = 0$ if and only if M is almost K-contact.*

Proof The defining relation of almost K-contact manifolds is $\nabla_\xi \phi = 0$, or equivalently $\nabla_\xi \Phi = 0$. Since φ is nearly parallel, for any vector field x,

$$\begin{aligned}(\nabla_\xi \phi)(x) &= \nabla_\xi(\phi x) - \phi(\nabla_\xi x) = \nabla_\xi(P(\xi, x)) - P(\xi, \nabla_\xi x) \\ &= P(\nabla_\xi \xi, x) + P(\xi, \nabla_\xi x) - P(\xi, \nabla_\xi x) = P(\nabla_\xi \xi, x),\end{aligned} \tag{3.24}$$

that is zero if and only if $\nabla_\xi \xi$ is zero. □

Theorem 4 *Let (ϕ, η, ξ, g) be an almost contact metric structure induced by a G_2 structure and $v = \sum_{i=1}^{6} P(e_i, \nabla_{e_i} \xi)$. If $g(\xi, v) \neq 0$, then $\nabla\Phi$ is not of classes $\mathcal{D}_1, \mathcal{C}_5, \mathcal{C}_7, \mathcal{C}_8, \mathcal{C}_9, \mathcal{C}_{10}, \mathcal{C}_{11}, \mathcal{C}_{12}$.*

Proof First, to compute $i_{10}(\nabla\Phi)$, we write

$$(\nabla_{e_i}\Phi)(e_i,\xi) = g(e_i,\nabla_{e_i}(P(\xi,\xi))) + g(\nabla_{e_i}\xi, P(\xi,e_i)) = g(P(e_i,\nabla_{e_i}\xi),\xi), \tag{3.25}$$

and we obtain

$$i_{10}(\nabla\Phi) = \sum_{i,j=1}^{6} g(P(e_i,\nabla_{e_i}\xi),\xi)g(P(e_j,\nabla_{e_j}\xi),\xi) = g^2(v,\xi). \tag{3.26}$$

Assume that $g(\xi,v)\neq 0$. Then $i_{10}(\nabla\Phi) = g(\xi,v)^2 \neq 0$ and the classes $\mathcal{D}_1,\mathcal{C}_5,\mathcal{C}_7,\mathcal{C}_8,\mathcal{C}_9,\mathcal{C}_{10},\mathcal{C}_{11},\mathcal{C}_{12}$ are eliminated, similar to previous proofs. □

Corollary 5 *If $g(\xi,v)\neq 0$ and $div(\xi)\neq 0$, then $\nabla\Phi$ is not an element of the classes $\mathcal{C}_i$, for $i=1,\cdots,12$.*

Next we give examples of a.c.m.s. induced by a calibrated G_2 structure ($d\varphi = 0$) and a nearly parallel G_2 structure, respectively. The a.c.m.s. induced by the calibrated G_2 structure is nearly cosymplectic and almost-K-contact, whereas that induced by the nearly parallel G_2 structure is almost-K-contact.

Example 6 *Let $\mathfrak{s}$ be the Lie algebra with structure equations*

$$de^1 = -\frac{1}{2}e^{17}, de^2 = -\frac{1}{2}e^{27}, de^3 = e^{37}, de^4 = e^{47},$$

$$de^5 = e^{13} - e^{24} - \frac{1}{2}e^{57}, de^6 = e^{14} + e^{23} - \frac{1}{2}e^{67}, de^7 = 0.$$

Then $\mathfrak{s}$ admits the calibrated G_2 structure

$$\varphi = -e^{136} + e^{145} + e^{235} + e^{246} + e^{567} - e^{127} - e^{347}$$

such that the metric g induced by φ is the one making the basis $\{e_1,\ldots,e_7\}$ orthonormal [8]. The cross product of frame elements can be written by using the identity (2.1). The nonzero brackets of frame elements are

$$[e_1,e_3] = -e_5, [e_1,e_4] = -e_6, [e_1,e_7] = \frac{1}{2}e_1, [e_2,e_3] = -e_6, [e_2,e_4] = e_5,$$

$$[e_2,e_7] = \frac{1}{2}e_2, [e_3,e_7] = -e_3, [e_4,e_7] = -e_4, [e_5,e_7] = \frac{1}{2}e_5, [e_6,e_7] = \frac{1}{2}e_6.$$

By Kozsul's formula, the nonzero covariant derivatives are

$$e_1 = 2\nabla_{e_1}e_7 = -2\nabla_{e_3}e_5 = -2\nabla_{e_4}e_6 = -2\nabla_{e_5}e_3 = -2\nabla_{e_6}e_4,$$

$$e_2 = 2\nabla_{e_2}e_7 = -2\nabla_{e_3}e_6 = 2\nabla_{e_4}e_5 = 2\nabla_{e_5}e_4 = -2\nabla_{e_6}e_3,$$

$$e_3 = 2\nabla_{e_1}e_5 = 2\nabla_{e_2}e_6 = -\nabla_{e_3}e_7 = 2\nabla_{e_5}e_1 = 2\nabla_{e_6}e_2,$$

$$e_4 = 2\nabla_{e_1}e_6 = -2\nabla_{e_2}e_5 = -\nabla_{e_4}e_7 = -2\nabla_{e_5}e_2 = 2\nabla_{e_6}e_1,$$

$$e_5 = -2\nabla_{e_1}e_3 = 2\nabla_{e_2}e_4 = 2\nabla_{e_3}e_1 = -2\nabla_{e_4}e_2 = 2\nabla_{e_5}e_7,$$

$$e_6 = -2\nabla_{e_1}e_4 = -2\nabla_{e_2}e_3 = 2\nabla_{e_3}e_2 = 2\nabla_{e_4}e_1 = 2\nabla_{e_6}e_7,$$

$$e_7 = -2\nabla_{e_1}e_1 = -2\nabla_{e_2}e_2 = \nabla_{e_3}e_3 = \nabla_{e_4}e_4 = -2\nabla_{e_5}e_5 = -2\nabla_{e_6}e_6.$$

Now we show that there is no almost cosymplectic ($d\eta = 0$ and $d\Phi = 0$) structure induced by φ on $\mathfrak{s}$. Let $\eta = \sum_{i=1}^{7} a_i e^i$ be any 1-form on $\mathfrak{s}$, where a_i are constants. By direct calculation $d\eta = 0$ iff $a_i = 0$ for $i = 1, \dots, 6$. Thus, to obtain an almost cosymplectic structure (ϕ, ξ, η, g) (such that $d\eta = 0$ and $d\Phi = 0$), one must have $\eta = e^7$ and $\xi = e_7$. In this case, since

$$\Phi(x, y) = g(x, \phi(y)) = g(x, P(e_7, y)) = -\varphi(e_7, x, y) = -(e_7 \lrcorner \varphi)(x, y),$$

the fundamental 2-form of the a.c.m.s. is $\Phi = e^{12} + e^{34} - e^{56}$. Since

$$d\Phi = e^{127} - e^{136} + e^{145} + e^{235} + e^{246} - 2e^{347} - e^{567} \neq 0,$$

there is no almost cosymplectic (in particular cosymplectic) structure induced by φ on $\mathfrak{s}$.

Consider the a.c.m.s. (ϕ, ξ, η, g) induced by φ on $\mathfrak{s}$, where $\eta = e^7$, $\xi = e_7$ and $\Phi = e^{12} + e^{34} - e^{56}$. Since

$$\nabla_{e_7}\Phi(e_i, e_j) = e_7[\Phi(e_i, e_j)] - \Phi(\nabla_{e_7}e_i, e_j) - \Phi(e_i, \nabla_{e_7}e_j) = 0,$$

this structure is almost-K-contact ($\nabla_\xi \Phi = 0$, or equivalently $\nabla_\xi \phi = 0$).

Since $(\nabla_{e_1}\Phi)(e_7, e_2) = -\frac{1}{2} \neq 0$, the defining relation of the class $\mathcal{D}_1$ is not satisfied; see the defining relation (2.2).

The a.c.m.s. is not in $\mathcal{D}_2$, since $(\nabla_{e_1}\Phi)(e_3, e_6) = -1$, whereas

$$\eta(e_1)(\nabla_{e_7}\Phi)(e_3, e_6) + \eta(e_3)(\nabla_{e_1}\Phi)(e_7, e_6) + \eta(e_6)(\nabla_{e_1}\Phi)(e_3, e_7) = 0;$$

see the relation (2.3). In addition, for $x = e_1$, $y = e_3$ and $z = e_6$, it can be checked that the defining relation of $\mathcal{C}_{12}$ is not satisfied; refer to (2.4).

An a.c.m.s. is called nearly cosymplectic if $\nabla_x \Phi(x, y) = 0$ for all vector fields x, y. Direct calculation yields $(\nabla_{e_i}\Phi)(e_j, e_k) + (\nabla_{e_j}\Phi)(e_i, e_k) = 0$ for all basis elements. Thus, the a.c.m.s is nearly cosymplectic.

Next consider the a.c.m.s. (ϕ, ξ, η, g) induced by φ, where $\xi = e_1$. Then $\eta = e^1$ and $\Phi = e^{27} + e^{36} - e^{45}$. Since $\nabla_\xi \Phi(\xi, e_2) = -\frac{1}{2}$, this structure is not cosymplectic, nearly cosymplectic, almost-K-contact, or an element of $\mathcal{D}_1$. Moreover, $\nabla_\xi \xi = \nabla_{e_1}e_1 \neq 0$, which implies by Theorem 1 that the structure is not in $\mathcal{D}_2$, $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{11}$. In addition, for $x = e_1$, $y = e_3$, and $z = e_4$, the defining relation of the class $\mathcal{C}_{12}$ is not satisfied; see the defining relation (2.4).

Example 7 *A Sasakian manifold is a normal contact metric manifold or equivalently an almost contact metric structure (ϕ, ξ, η, g) such that*

$$(\nabla_x \phi)(y) = g(x, y)\xi - \eta(y)x;$$

see [5]. In addition, the following properties are satisfied for all vector fields x, y:

$$\nabla_x \xi = -\phi(x), \qquad d\eta(x, y) = 2g(x, \phi(y)).$$

A 7-dimensional 3-Sasakian manifold is a Riemannian manifold (M,g) *equipped with three Sasakian structures* (ϕ_i,ξ_i,η_i,g), $i=1,2,3$ *satisfying*

$$[\xi_1,\xi_2]=2\xi_3,\quad [\xi_2,\xi_3]=2\xi_1,\quad [\xi_3,\xi_1]=2\xi_2$$

and

$$\phi_3\circ\phi_2=-\phi_1+\eta_2\otimes\eta_3,\quad \phi_2\circ\phi_3=\phi_1+\eta_3\otimes\eta_2,$$
$$\phi_1\circ\phi_3=-\phi_2+\eta_3\otimes\eta_1,\quad \phi_3\circ\phi_1=\phi_2+\eta_1\otimes\eta_3,$$
$$\phi_2\circ\phi_1=-\phi_3+\eta_1\otimes\eta_2,\quad \phi_1\circ\phi_2=\phi_3+\eta_2\otimes\eta_1.$$

The vertical subbundle T^v *is spanned by* ξ_1, ξ_2, *and* ξ_3. *Both* T^v *and its orthogonal complement* $T^h=$ *span*$\{e_4,e_5,e_6,e_7\}$ *are invariant under* ϕ_i. *There exists a local orthonormal frame* $\{e_1,\cdots,e_7\}$ *such that* $e_1=\xi_1$, $e_2=\xi_2$, *and* $e_3=\xi_3$ *and the endomorphisms* ϕ_i *acting on the horizontal bundle are given by the matrices below:*

$$\phi_1:=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix},\quad \phi_2:=\begin{bmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{bmatrix},\quad \phi_3:=\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}.$$

The corresponding coframe via the Riemannian metric is denoted by $\{\eta_1,\cdots,\eta_7\}$. *The differentials* $d\eta_i$, $i=1,2,3$ *are*

$$d\eta_1=-2(\eta_{23}+\eta_{45}+\eta_{67}),\quad d\eta_2=2(\eta_{13}-\eta_{46}+\eta_{57}),\quad d\eta_3=-2(\eta_{12}+\eta_{47}+\eta_{56}).$$

Consider the 3-form

$$\varphi=\frac{1}{2}\eta_1\wedge d\eta_1-\frac{1}{2}\eta_2\wedge d\eta_2-\frac{1}{2}\eta_3\wedge d\eta_3$$

constructed in [1]. This G_2 *structure is one of three nearly parallel* G_2 *structures given in Theorem 6.2 in [1]. We denote* φ_1 *in [1] by* φ.

Now we give an example of an almost contact metric structure on a 3-Sasakian manifold with the nearly parallel G_2 *structure* φ. *Definitions, endomorphisms given as matrices, differentials, and the* G_2 *structure* φ *can be found in [1].*

By Kozsul's formula we obtain $\nabla_{e_i}e_i=0$ *for* $i=1,2,3$ *and*

$$\nabla_{e_1}e_2=e_3,\nabla_{e_1}e_3=-e_2,\nabla_{e_2}e_1=-e_3,\nabla_{e_2}e_3=e_1,\nabla_{e_3}e_1=e_2,\nabla_{e_3}e_2=-e_1.$$

By the local expression of

$$\begin{aligned}\varphi&=\tfrac{1}{2}\eta_1\wedge d\eta_1-\tfrac{1}{2}\eta_2\wedge d\eta_2-\tfrac{1}{2}\eta_3\wedge d\eta_3\\&=\eta_{123}-\eta_{145}-\eta_{167}+\eta_{246}-\eta_{257}+\eta_{347}+\eta_{356},\end{aligned}$$

the 2-fold vector cross products of frame elements are computed by equation (2.1).

Consider the a.c.m.s. (ϕ,ξ,η,g) *on* M *induced by the 2-fold vector cross product of the nearly parallel* G_2 *structure* φ, *where* $\xi=e_1=\xi_1$, $\eta=\eta_1$ *and* $\phi(x)=P(\xi,x)$. *First, since*

$$(\nabla_x\Phi)(y,z)=g(y,\nabla_x(P(e_1,z)))+g(\nabla_x z,P(e_1,y)),$$

we have $(\nabla_{e_2}\Phi)(e_1,e_2) = 1 \neq 0$ *and thus the a.c.m.s. is not cosymplectic and not in* $\mathcal{D}_1 = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4$*; see (2.2).*

Moreover, the a.c.m.s. is not semi-cosymplectic ($\delta\eta = 0$ *and* $\delta\Phi = 0$*). To see this, we compute*

$$\delta\Phi(e_1) = -\sum_{i=1}^{7}(\nabla_{e_i}\Phi)(e_i,e_1) = \sum_{i=1}^{7}\Phi(e_i,\nabla_{e_i}e_1) = -\sum_{i=1}^{7}g(\nabla_{e_i}e_1, P(e_1,e_i)).$$

Note that $\nabla_{e_i}e_1 = -\phi_1(e_i)$*. Thus, we obtain* $\delta\Phi(e_1) = -2$*.*

In addition, the a.c.m.s. is not trans-Sasakian; that is, the defining relation

$$\begin{aligned}(\nabla_x\Phi)(y,z) \quad = \quad & -\frac{1}{2n}\{(g(x,y)\eta(z) - g(x,z)\eta(y))\delta\Phi(\xi) \\ & +(g(x,\phi(y))\eta(z) - g(x,\phi(z))\eta(y))\delta\eta\}\end{aligned} \tag{3.27}$$

is not satisfied. For $x = e_2$*,* $y = e_1$*,* $z = e_2$*, the left-hand side of the equation (3.27) is*

$$(\nabla_{e_2}\Phi)(e_1,e_2) = 1,$$

whereas the right-hand side is

$$\frac{1}{3}\{g(e_2,e_1)\eta(e_2) - g(e_2,e_2)\eta(e_1)\} = -\frac{1}{3}.$$

In particular, the a.c.m.s. is not α*-Sasakian or* β*-Kenmotsu. Note that we started with a Sasakian structure on a manifold and then we used the 2-fold vector cross product of the nearly parallel* G_2 *structure* φ*; however, the induced a.c.m.s. is not Sasakian.*

Note that since $\nabla_\xi\xi = 0$*, the a.c.m.s. is almost-K-contact by Theorem 3. It can be seen that for* $\xi = ae_1 + be_2 + ce_3$*, where* a*,* b*,* c *are constants, one has* $\nabla_\xi\xi = 0$*. Therefore, by Theorem 3, the a.c.m.s. where* $\xi = ae_1 + be_2 + ce_3$ *is almost-K-contact.*

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