REEL CLİFFORD CEBİRLERİNİN TEMSİLLERİ

Şenay KARAPAZAR
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ÖZET

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Danışman: Yrd.Doç.Dr. Nedim DEĞİRMENCİ

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Bu tezde reel Clifford cebirleri ve temsilleri incelenmiştir. İlk iki bölümde Clifford cebirlerinin tanımlanmasında kullanılan simetrik bi-lineer formlar ve tensör çarpımları verilmiştir. Üçüncü bölümde sonlu boyutlu reel vektör uzayları üzerinde Clifford cebirleri tanımlanmış ve bazı özellikleri elde edilmiştir. Dördüncü bölümde $\mathbb{R}^n$ üzerindeki dejenere olmayan Clifford cebirlerinin izomorf olduğu matris cebirlerinin tablosu verilmiştir. Bu tablo genel olarak bilinmesine rağmen sözkonusu izomorfizmlerin açık ifadeleri literatürde bulunmamaktadır. Beşinci bölümde sözkonusu tabloda geçen izomorfizmlerin açık ifadeleri verilmiş ve genel olarakta $\mathbb{R}^n$ üzerindeki dejenere olmayan bir Clifford cebirlerinden izomorf olduğu matris cebrine giden izomorfizmin nasıl elde edileceğine dair bir yöntem verilmiştir. Altıncı ve yedinci bölümlerde dejenere olmayan Clifford cebirlerinin temsilleri verilmiştir. Son bölümde de dejenere Clifford cebirlerine değinilmiştir.

Anahtar Kelimeler: Simetrik bi-lineer form, Kuadratik form, Tensör çarpımı, Clifford Cebri, Cebirin Temsili

ABSTRACT

Master of Science Thesis

REPRESANTATİONS OF REAL CLİFFORD ALGEBRAS

Şenay KARAPAZAR

Anadolu University

Graduate School of Natural and Applied Sciences

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Supervisor: Yrd.Doç.Dr. Nedim DEĞİRMENCİ

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In this thesis Real Clifford algebras and their represantations are studied. In the first two chapters symmetric bilinear forms and tensor product which are necessary to define Clifford algebras are given. In the third chapter Clifford algebras on the finite-dimensional real vector spaces are defined and their some properties are obtained. In the fourth chapter the table of matrix algebras which become isomorphic of non-degenerate Clifford algebras on $\mathbb{R}^n$ is given. Although this table was generally known, there are no explicit expressions of these isomorphisms in literature. In the fifth chapter, Clear expressions of the isomorphisms in the mentioned on table is given and thus a method is developed to obtain an isomorphism from a non-degenerate Clifford algebra on $\mathbb{R}^n$ to related matrix algebra. In the sixth and seventh chapter represantations of non-degenerate Clifford algebras were given. In the last chapter degenerate Clifford algebras are mentioned.

Keywords: Symmetric bilinear form, Quadratic form, Tensor product, Clifford algebra, Represantation of algebra

TEŞEKKÜR

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İÇİNDEKİLER

SİMGELER DİZİNİ

$\cong$	İzomorf
$\otimes$	Tensör çarpımı
$L(E;F)$	E den F ye lineer dönüşüm
$T(V)$	Tensör cebri
$Cl(V,Q)$	Clifford cebri
$\hat{\otimes}$	Graded tensör çarpımı
$Cl_{p,q}$	Dejenere olmayan Clifford cebri
$Cl_{p,q,r}$	Dejenere Clifford cebri
$O(V,b)$	Ortogonal grup
(V,b)	Bi-lineer uzay
(V,Q)	Kuadratik uzay
$\Lambda(V)$	V nin dış cebri
$\blacksquare$	Kanıtın sonu

1 SİMETRİK Bİ-LİNEER FORMLAR

Bu bölümde, Clifford cebirlerinin tanımlanmasında temel unsurlardan biri olan simetrik bi-lineer formların bazı temel özelliklerine değineceğiz. Bunları sonlu boyutlu reel vektör uzayları için göz önüne alacağız.

Tanım 1.1 *V sonlu boyutlu reel vektör uzayı olsun ve $b : V \times V \longrightarrow \mathbb{R}$ dönüşümü verilsin. Eğer b dönüşümü her $x, y, x', y' \in V$ ve her $\alpha \in \mathbb{R}$ için,*

$$\begin{aligned} b(x + x', y) &= b(x, y) + b(x', y) \\ b(x, y + y') &= b(x, y) + b(x, y') \\ b(\alpha x, y) &= \alpha . b(x, y) \\ b(x, \alpha y) &= \alpha . b(x, y) \end{aligned}$$

koşullarını sağlıyorsa bu dönüşüme V üzerinde bir bi-lineer (iki lineer) form denir. Ayrıca $\forall x, y \in V$ için $b(x, y) = b(y, x)$ koşulu da sağlanıyorsa b ye simetrik bi-lineer form denir. Bu durumda (V, b) ikilisine de simetrik bi-lineer uzay denir.

Tanım 1.2 *(V, b) simetrik bi-lineer uzay olmak üzere, $x, y \in V$ için*

$$b(x, y) = b(y, x) = 0$$

oluyorsa x ve y vektörlerine ortogonal (dik) vektörler denir. Ortogonallik kavramı alt-kümeler için de anlamlıdır. Yani X ve Y, V nin alt kümeleri iken her $x \in X$ ve her $y \in Y$ için $b(x, y) = 0$ oluyorsa X ve Y ye ortogonaldirler denir.

Yardımcı Teorem 1.1 *$X^{\perp} = \{x \in V \mid$ her $y \in X$ için $b(x, y) = 0 \}$ alt kümesi V nin bir alt uzayıdır.*

Kanıt. $y, y' \in X^{\perp}$ alalım. $\forall x \in X$ için $b(y, x) = b(y', x) = 0$ dir.

$$b(y + y', x) = b(y, x) + b(y', x) = 0$$

olacağından $y + y' \in X^{\perp}$ elde edilir. $\alpha \in \mathbb{R}$ olsun. $\forall x \in X$ için

$$\alpha b(y, x) = 0 \Longrightarrow b(\alpha y, x) = 0 \Longrightarrow \alpha y \in X^{\perp}$$

olur. Buradan $X^{\perp}$ in, V nin bir alt uzayı olduğu elde edilir. ■

$X \subset Y$ ise $Y^{\perp} \subset X^{\perp}$ dir. $y \in Y^{\perp}$ alalım. $x \in X \subset Y$ için $b(x,y) = 0$ dır. $x \in X$ ve $b(x,y) = 0$ olduğundan $y \in X^{\perp}$ olmalıdır. Buradan $Y^{\perp} \subset X^{\perp}$ sonucu elde edilir. $W \subset V$ ise $(W, b|_{w\times w})$ bir simetrik bi-lineer uzaydır ve bunu $b|_{w\times w} = b_w$ ile göstereceğiz.

Tanım 1.3 *(V,b) ve (V',b') simetrik bi-lineer uzaylar olsun.*

(i) Eğer 1-1 bir $\sigma : V \longrightarrow V'$ lineer dönüşümü $\forall x, y \in V$ için

$$b'(\sigma(x), \sigma(y)) = b(x,y)$$

koşulunu sağlıyorsa bir izometri olarak adlandırılır.

(ii) $\sigma : V \longrightarrow V'$ örten bir izometri varsa (V,b) ve (V',b') uzayları izometrik veya izomorfik olarak adlandırılır ve $(V,b) \cong (V',b')$ olarak gösterilir.

(iii) $\sigma : V \longrightarrow V$ izometrilerin oluşturduğu gruba ortogonal grup veya (V,b) nin otomorfizm grubu denir. $b(\sigma(x), \sigma(y)) = b(x,y)$ gösterimi $\sigma^* b = b$ ile ifade edilebilir. $O(V,b)$ veya $Aut(V,b)$ ile gösterilir. Ortogonal grubun elemanları ortogonal dönüşümlerdir.

Tanım 1.4 *V bir vektör uzayı olsun. $Q : V \longrightarrow \mathbb{R}$ dönüşümü,*

i) $\forall x \in V, \alpha \in \mathbb{R}$ için $Q(\alpha x) = \alpha^2 Q(x)$ dir.

ii) $\forall x, y \in V$ için $b(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$ dönüşümü V üzerinde simetrik bi-lineer formdur

koşullarını sağlıyorsa, bu dönüşüme V üzerinde bir kuadratik form denir. Bu durumda (V,Q) çifti bir kuadratik uzay olarak adlandırılır.

ii)-de ki b simetrik bi-lineer formuna Q ya karşılık gelen simetrik bi-lineer form denir. Diğer taraftan V üzerinde bir b simetrik bi-lineer formu verildiğinde $Q(x) = b(x,x)$ şeklinde tanımlı $Q : V \to \mathbb{R}$ dönüşümü V üzerinde bir kuadratik formdur ve bu kuadratik forma b ye karşılık gelen kuadratik form denir.

1.1 Simetrik bi-lineer formlar için matris gösterimi

(V,b) simetrik bi-lineer uzay ve $\xi = \{e_1, e_2, ..., e_n\}$ V nin bir tabanı olsun. $B = (b(e_i, e_j)) = (b_{ij})$ şeklinde tanımlanan B matrisine b bi-lineer formunun ξ tabanına göre matrisi denir.

Herhangi $x, y \in V$ vektörlerinin b altındaki görüntülerini B matrisi yardımıyla hesaplamakta mümkündür. Şöyle ki $x = \sum_{i=1}^{n} x_i e_i$ ve $y = \sum_{j=1}^{n} y_j e_j$ olmak üzere,

$$\begin{aligned} b(x,y) &= b\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \sum_{i,j} x_i b(e_i, e_j) y_j = \sum_{i,j} x_i b_{ij} y_j \\ &= \sum_i x_i (B.y)_i = x^t B y \end{aligned}$$

elde edilir. V nin başka bir $\xi' = \{e'_1, e'_2, ..., e'_n\}$ tabanı için, $T = (t_{ij})$ taban değişim matrisi olmak üzere $j = 1, ..., n$ için $e'_j = \sum_{i=1}^{n} t_{ij} e_i$ yazabiliriz. Şimdi ξ' tabanına karşılık gelen b bi-lineer formunun matrisini hesaplayacağız.

Yardımcı Teorem 1.2 $B_{b,\xi'} = T^t B_{b,\xi} T$ *dir.*

Kanıt.

$$\begin{aligned} b(e'_i, e'_j) &= b\left(\sum_{k=1}^{n} t_{ki} e_k, \sum_{l=1}^{n} t_{lj} e_l\right) = \sum_{k,l} t_{ki} b_{kl} t_{lj} \\ &= \sum_k t_{ki} (B.T)_{kj} = (T^t B T)_{ij} \end{aligned}$$

■

Eğer $B = T^t A T$ olacak şekilde terslenebilir T matrisi varsa A ve B matrislerine benzerdirler denir.

Teorem 1.1 *İki bi-lineer uzay izometriktir* $\Longleftrightarrow$ *iki bi-lineer uzayın simetrik bi-lineer formlarının keyfi tabanlara göre belirttikleri matrisler benzerdir.*

Kanıt. ($\Longrightarrow$) *(V, b) ve (V', b') sırasıyla $\xi = \{e_1, e_2, ..., e_n\}$, $\xi' = \{e'_1, e'_2, ..., e'_n\}$ tabanları ile iki bi-lineer uzay ve $(V, b) \cong (V', b')$ olsunlar. O zaman $\sigma : V \longrightarrow V'$ birebir örten lineer dönüşümü vardır .öyle ki $\sigma(e_i) = \sum_{j=1}^{m} e'_j S_{ji}$ ile $S = (S_{ji})_{m \times n}$ matrisi tanımlar. σ dönüşümü izometri olduğundan*

$$x^t B y = b(x, y) = b'(\sigma(x), \sigma(y)) = (\sigma(x))^t B'(\sigma(y))$$

dir. $x = \sum_{i=1}^{n} x_i e_i$ ve $y = \sum_{j=1}^{n} y_j e_j$ ise $\sigma(x) = S \cdot x$ ve $\sigma(y) = S \cdot y$ dir. O halde

$$x^t B y = (\sigma(x))^t B'(\sigma(y)) = (S \cdot x)^t B'(S \cdot y) = x^t S^t B' S y$$

eşitliğinden $B = S^t B' S$ sonucu elde edilir.

($\Longleftarrow$) (V,b) *nin* $\xi = \{e_1, e_2, ..., e_n\}$ *tabanına göre matrisi* B *ve* (V', b') *nin* $\xi' = \{e'_1, e'_2, ..., e'_n\}$ *tabanına göre matrisi* B' *olsun.* (V,b) *ve* (V', b') *bi-lineer uzaylarının bu tabanlara göre belirttikleri simetrik matrisler benzer olsun. Yani* $B = S^t B' S$ *olacak şekilde* S *terslenebilir matrisi var olsun.* $\sigma : V \longrightarrow V'$ *birebir örten lineer dönüşümünü her* $x \in V$ *için* $\sigma(x) = S \cdot x$ *şeklinde tanımlayalım. O zaman* $x = \sum_{i=1}^{n} x_i e_i$ *ve* $y = \sum_{j=1}^{n} y_j e_j$ *ise* $\sigma(x) = S \cdot x$ *ve* $\sigma(y) = S \cdot y$ *dir.*

$$\begin{aligned} b(x,y) &= x^t B y = x^t S^t B' S y = (S\cdot x)^t B' (S \cdot y) \\ &= (\sigma(x))^t B' (\sigma(y)) = b'(\sigma(x), \sigma(y)) \end{aligned}$$

olduğundan iki bi-lineer uzay izometriktir. ■

Özel olarak $(V,b) = (V', b')$ ve $\xi = \xi'$ ise izometrinin S matrisi için $B = S^t B S$ koşulu elde edilir.

$O(V,b) \cong \{S \mid \det S \neq 0, B = S^t B S\}$ izomorfizmi vardır. $\sigma : V \longrightarrow V$ birebir örten lineer dönüşüm ve $\sigma(e_i) = \sum_{k=1}^{n} S_{ki} e_k$ olmak üzere,

$$\begin{aligned} b(e_i, e_j) &= b(\sigma(e_i), \sigma(e_j)) \\ &= b\left(\sum_{k=1}^{n} S_{ki} e_k, \sum_{l=1}^{n} S_{lj} e_l\right) = \sum_{k,l} S_{ki} b(e_k, e_l) S_{lj} \\ &= \sum_{k,l} S_{ki} b_{kl} S_{lj} = \sum_{k,l} S_{ki} (BS)_{kj} = (S^t B S)_{ij} \end{aligned}$$

olduğundan $B = S^t B S$ sonucu elde edilir.

Böylece simetrik bi-lineer uzayların izometri sınıfları ile simetrik matrislerin benzerlik sınıfları arasında birebir örten bir eşleme vardır.

V bir vektör uzayı ve V^*, V nin dual uzayını belirtsin.

$$V^* = \{f \mid f : V \to \mathbb{R} \text{ lineer}\}$$

uzayı, tüm lineer fonksiyonellerin vektör uzayıdır. V nin tabanı $\{e_1, e_2, ..., e_n\}$ ise buna karşılık gelen $\{e_1^*, e_2^*, ..., e_n^*\}$, V^* dual uzayının tabanı olarak adlandırılır ve

$$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, i = j \text{ iken} \\ 0, i \neq j \text{ iken} \end{cases}$$

olarak tanımlanır. Ayrıca $x = \sum_{j=1}^{n} x_j e_j \in V$ için

$$e_i^*(x) = e_i^*\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j e_i^*(e_j) = x_i$$

olacağından herhangi bir $x \in V$ için $e_i^*(x) = x_i$ dir.

Tanım 1.5 *(V, b) bi-lineer uzay ise*

$$\begin{aligned} \hat{b}: \quad V &\rightarrow V^* \\ x &\mapsto \hat{b}(x): \; V \rightarrow \mathbb{R} \\ &\qquad\qquad y \mapsto \left(\hat{b}(x)\right)(y) \end{aligned}$$

$\left(\hat{b}(x)\right)(y) = b(x, y)$ olarak tanımlanan $\hat{b}$ lineer dönüşümüne adjoint dönüşümü denir.

Yardımcı Teorem 1.3 *$\xi = \{e_1, e_2, ..., e_n\}$ V nin tabanı ve buna karşılık gelen $\xi^* = \{e_1^*, e_2^*, ..., e_n^*\}$ V^* dual uzayının tabanı ise ξ ve ξ^* tabanlarına karşılık gelen $\hat{b}$ nın matrisi, ξ tabanına karşılık gelen b bi-lineer uzayının B matrisidir.*

Kanıt. *$\left(\hat{b}(e_i)\right)(e_j) = b(e_i, e_j)$ olduğu kullanılırsa,*

$$\hat{b}(e_i) = \sum_{k=1}^{n} x_k e_k^* \Longrightarrow \left(\hat{b}(e_i)\right)(e_j) = \left(\sum_{k=1}^{n} x_k e_k^*\right)(e_j) = \sum_{k=1}^{n} x_k e_k^*(e_j)$$

$j = k$ iken $\left(\hat{b}(e_i)\right)(e_j) = x_j$ olacaktır. Buradan $\hat{b}(e_i) = \sum_{k=1}^{n} b(e_i, e_k) e_k^*$ elde edilir. O halde $\hat{b}$ nın matrisi, ξ tabanına karşılık gelen b bi-lineer uzayının B matrisidir. ■

Yardımcı Teorem 1.4 *W, (V, b) bi-lineer uzayının bir alt uzayı ise $\pi : V^* \longrightarrow W^*$ projeksiyon olmak üzere $W^\perp = Ker\left(\pi \circ \hat{b}\right)$ dir.*

Kanıt. *$x \in W^\perp$ alalım. $y \in W$ için $\left(\hat{b}(x)\right)(y) = b(x, y) = 0$ dır. Buradan $\left(\hat{b}(x)\right)|_W = 0$ olacağından $x \in Ker\left(\pi \circ \hat{b}\right)$ elde edilir. O halde $W^\perp \subset Ker\left(\pi \circ \hat{b}\right)$ dir(1). Şimdi $z \in Ker\left(\pi \circ \hat{b}\right)$ olsun.*

$$\left(\pi \circ \hat{b}\right)(z) = 0 \Longrightarrow \left(\hat{b}(z)\right)|_W = 0$$

$y \in W$ için $b(z, y) = 0$ olduğundan $z \in W^\perp$ elde edilir ve buradan $Ker\left(\pi \circ \hat{b}\right) \subset W^\perp$ (2) olur. Böylece (1) ve (2) den $W^\perp = Ker\left(\pi \circ \hat{b}\right)$ sonucu elde edilir. ■

1.2 Regüler Uzaylar ve Ortogonal Toplam

(V, b) simetrik bi-lineer uzay olsun. $V^{\perp} = \{0\}$ ise V ye regülerdir, non-singülerdir yada non-dejeneredir denir. Buna göre regüler bir uzayda $x \in V$ olmak üzere her $y \in V$ için $b(x, y) = 0$ ise $x = 0$ dır, yani regüler bir uzayda tüm vektörlere dik olan yegane vektör sıfır vektörüdür. $V^{\perp}$ alt uzayı (V, b) simetrik bi-lineer uzayının radikali olarak adlandırılır ve

$$Rad(V) = V^{\perp} = \{x \in V \mid \text{her } y \in V \text{ için } b(x, y) = 0\}$$

ile tanımlanır.

Sonuç 1.1 *$\hat{b}$ bir izomorfizm (veya buna denk olarak b bi-lineer formunun matrisi B terslenebilir) ise (V, b) regülerdir.*

Kanıt. $\hat{b}$ bir izomorfizm ise (V, b) nin regüler olduğunu gösterelim. $v \in V^{\perp}$ alalım. $\forall x \in V$ için $b(x, v) = 0$ dır. $\left(\hat{b}(v)\right)(x) = 0$ ve $\hat{b}$ lineer olduğundan $\hat{b}(v) = 0$ dır. $\hat{b}(v) = 0 = \hat{b}(0)$ ve $\hat{b}$ birebir olduğundan $v = 0$ olmalıdır. O halde (V, b) regülerdir. Şimdi de b nin matrisi B terslenebilir olduğunda (V, b) nin regüler olduğunu gösterelim. $v \in V^{\perp}$ alalım. $\forall x \in V$ için $b(x, v) = 0$ dır. $b(x, v) = x^t B v = 0$ olduğundan ve bu her x için sağlanacağından $Bv = 0$ olmalıdır. B terslenebilir olduğundan $B^{-1}Bv = B^{-1}0$ eşitliğinden $v = 0$ olur. O halde (V, b) regülerdir. ■

Tanım 1.6 *$V_1, V_2, \ldots, V_k$ lar V nin alt uzayları olsunlar. Her bir $x \in V$ elemanı $x_i \in V_i$ olmak üzere $x = \sum_{i=1}^{k} x_i$ şeklinde tek türlü yazılabiliyorsa, V ye $V_1, V_2, \ldots, V_k$ alt-uzaylarının bir iç direkt toplamıdır denir ve $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ şeklinde gösterilir.*

Tanım 1.7 *V_1 ve V_2, (V, b) uzayının iki alt uzayı olsun. Eğer $V = V_1 \oplus V_2$ ve $V_1 \perp V_2$ ise V ye V_1 ve V_2 nin ortogonal toplamı denir. V_1 ve V_2 ye V nin tamamlayıcı alt uzayları da denir ve $V = V_1 \perp V_2$ ile gösterilir. Genel olarak $V = V_1 \oplus \ldots \oplus V_n$ ve $i \neq j$ için $V_i \perp V_j$ ise $V = V_1 \perp \ldots \perp V_n$ dir. Böyle bir ayrışıma (V, b) nin ortogonal ayrışımı denir.*

Yardımcı Teorem 1.5 W, (V, b) *nin regüler alt uzayı ise* $V = W \perp W^{\perp}$ *dir.*

Kanıt. *$W \perp W^{\perp}$ olduğu açıktır. Bu yüzden $V = W \oplus W^{\perp}$ olduğunu göstermek yetecektir. W regüler olduğundan ve yardımcı teorem (1.4) den $Ker\left(\hat{b}_w\right) = 0$ dır. $x \in V$ ve $f = \left(\hat{b}(x)\right)|_w$ ise W regüler olduğundan $f = \left(\hat{b}_w\right)(y)$ olacak şekilde bir $y \in W$ vardır. Böylece her $z \in W$ için,*

$$b(x,z) = \left(\hat{b}(x)\right)(z) = f(z) = \left(\hat{b}_w(y)\right)(z) = b(y,z) \qquad (1.1)$$

olacağından $b(x,z) - b(y,z) = 0, b(x-y,z) = 0$ elde edilir. $z \in W$ olduğundan $x - y \in W^{\perp}$ olur. (1.1) denkleminden dolayı $x = y + (x-y)$ yazabiliriz. Bu da $V = W + W^{\perp}$ olduğunu ispatlar. ■

Teorem 1.2 *Her (V, b) simetrik bi-lineer uzayı bir boyutlu alt uzayların ortogonal toplamıdır. Diğer bir ifadeyle V, ortogonal vektör çiftlerinden oluşan bir tabana sahiptir.*

Kanıt. $b = 0$ ise V nin her ayrışımı direkt toplam olarak ortogonal ayrışımdır. $b \neq 0$ ise $b(x,y) \neq 0$ olacak şekilde $x, y \in V$ vardır.

$$b(x,y) = \frac{1}{2}\left(b(x+y, x+y) - b(x,x) - b(y,y)\right)$$

olduğundan $b(z,z) \neq 0$ olacak şekilde bir z vektörü var olmalıdır($z = x, y, x+y$ gibi). $V_1 = \mathbb{R}z$ bir boyutlu alt uzayı regülerdir. Önceki yardımcı teoremden $V = V_1 \perp V_1^{\perp}$ dir. $V_1^{\perp}$ e tümevarım yönteminin uygulanması ile kanıt tamamlanır. ■

$\mathbb{R}$ deki katsayılarla sütun vektörlerinin $\mathbb{R}^n$ vektör uzayını düşünelim. B, $n \times n$ tipinde simetrik bir matris ise,

$$\begin{aligned} b_B : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (x,y) &\mapsto b_B(x,y) = x^t B y \end{aligned}$$

şeklinde tanımlanan b_B simetrik bi-lineer formdur ve $(\mathbb{R}^n, b_B)$ bi-lineer uzayını $< B >$ ile göstereceğiz.

Teorem 1.3 *Her simetrik B matrisi bir diagonal matrise benzerdir.*

Kanıt. Önceki teoremden simetrik $\langle B \rangle$ bi-lineer uzayı için $\{x_1, \ldots, x_n\}$ ortogonal tabanını seçelim. Bu durumda $(b_B(x_i, x_j))$ bir diagonal matristir ve $i \neq j$ iken $b_B(x_i, x_j) = 0$ dır. O halde bi-lineer uzayın matrisi aşağıdaki şekildedir.

$$\begin{bmatrix} b_B(x_1, x_1) & 0 & \cdots & 0 \\ 0 & b_B(x_2, x_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & b_B(x_n, x_n) \end{bmatrix}$$

■

Teorem 1.4 *(Sylvester) (V, b), n-boyutlu simetrik bi-lineer uzay olsun. V nin öyle bir $\{e_1, e_2, \cdots, e_n\}$ tabanı vardır ki b formunun bu tabana göre matrisi aşağıdaki şekildedir.*

$$B = \begin{bmatrix} I_p & & \\ & -I_q & \\ & & 0_r \end{bmatrix}$$

Burada $I_p, p \times p$ tipindeki birim matrisi, $I_q, q \times q$ tipindeki birim matrisi ve 0_r ise $r \times r$ tipindeki sıfır matrisini göstermektedir. ~~*Üstelik p, q sayıları seçilen tabandan bağımsızdır.*~~

Kanıt. Önceki teoremdeki $\{x_1, \ldots, x_n\}$ ortogonal tabanının elemanlarını $b(x_i, x_i) > 0$, $b(x_i, x_i) < 0$ ve $b(x_i, x_i) = 0$ olarak gruplandıralım. $b(x_i, x_i) > 0$ olanların sayısını p, $b(x_i, x_i) < 0$ olanların sayısını q ve $b(x_i, x_i) = 0$ olanların sayısını da r ile gösterelim. Şimdi de önce pozitif olanları, sonra negatif olanları son olarakta sıfır olanları yazmak suretiyle verilen taban elemanlarının yerlerini değiştirerek elde edilen yeni tabanı $\{z_1, \ldots, z_n\}$ şeklinde gösterelim.

$$1 \leq i \leq p \text{ için } e_i = \frac{1}{\sqrt{b(z_i, z_i)}} z_i$$
$$p + 1 \leq i \leq p + q \text{ için } e_i = \frac{1}{\sqrt{-b(z_i, z_i)}} z_i$$
$$p + q + 1 \leq i \leq p + q + r = n \text{ için } e_i = z_i$$

vektörlerini alalım. Elde edilen $\{e_1, e_2, \cdots, e_n\}$ vektörlerinin lineer bağımsızlığı z_i lerin lineer bağımsızlığından elde edilir. Ayrıca $\{e_1, e_2, \cdots, e_n\}$ tabanına göre b nin matrisi istenilen şekilde olur. ■

Üstelik buradaki p, q sayıları seçilen tabandan bağımsızdır (Bakınız [1]). Tek türlü belirli olan (p, q) çiftine b simetrik bi-lineer formunun imzası (signature) denir.

Yukarıdaki teoremde sözü edilen tabana (V, b) uzayının *Sylvester tabanı* denir. b nin bu tabana göre ifadesi

$$b(x, y) = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{q+p}$$

şeklinde olur. b ye karşılık gelen Q kuadratik formunun bu tabana göre ifadesi de

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

şeklindedir.

Tanım 1.8 *(V, b) ve (V', b') iki bi-lineer uzay ise $V \oplus V'$ vektör uzaylarının direkt toplamı altında simetrik bi-lineer uzayını $(V, b) \perp (V', b')$ olarak yazabiliriz. $(V, b) \perp (V', b')$ bi-lineer uzayı, (V, b) ve (V', b') bi-lineer uzaylarının ortogonal toplamı olarak adlandırılır. $b \perp b'$ bi-lineer formu da*

$$(b \perp b')\left((x, x'), (y, y')\right) = b(x, y) + b'(x', y')$$

şeklinde tanımlanır. Buradan sonlu tane bi-lineer uzayın ortogonal toplamını da tanımlayabiliriz. Ortogonal toplamların yapısı, doğal olarak birleşmelidir. Buna göre b ve b' ye karşılık gelen Q ve Q' kuadratik formların toplamı

$$(Q \oplus Q')(x, x') = Q(x) + Q'(x')$$

da $V \oplus V'$ üzerindeki $(b \perp b')$ simetrik bi-lineer formuna karşılık gelen kuadratik form olur.

2 TENSÖR ÇARPIMI

U ve V iki reel vektör uzayı ve W da $U \times V$ kümesi tarafından üretilen serbest vektör uzayı olsun. Yani tüm formal lineer kombinasyonların kümesi olsun.

$$F(U,V) = W = \left\{ \sum_{i \in I} a_i (u_i, v_i) \mid I \text{ sonlu, } a_i \in R, (u_i, v_i) \in U \times V \right\}$$

$x, y \in F(U,V) = W$ ve $c \in \mathbb{R}$ alalım. $x = \sum_{i=1}^{n} a_i (u_i, v_i)$ ve $y = \sum_{j=1}^{m} b_j (u'_j, v'_j)$ olsun.

$$\begin{aligned} x + y &= \sum_{i=1}^{n} a_i (u_i, v_i) + \sum_{j=1}^{m} b_j (u'_j, v'_j) \\ c.x &= \sum_{i=1}^{n} c.a_i (u_i, v_i) \end{aligned}$$

işlemleri ile $F(U,V) = W$ bir reel vektör uzayıdır.

$\mathcal{R} \subseteq W$ aşağıdaki tipdeki elemanlar tarafından üretilen alt uzay olsun.

$$\left. \begin{aligned} &(au + bu', v) - a(u,v) - b(u',v) \\ &(u, av + bv') - a(u,v) - b(u,v') \end{aligned} \right\} a, b \in \mathbb{R},\ u, u' \in U \text{ ve } v, v' \in V$$

$F(U,V)/\mathcal{R}$ bölüm uzayına U ve V nin tensör çarpımı denir ve $U \otimes V$ şeklinde gösterilir. Yani $U \otimes V = F(U,V)/\mathcal{R}$ dir. $u \in U, v \in V$ iken (u,v) nin doğal izdüşüm altındaki görüntüsü (yani (u,v) nin denklik sınıfı) $u \otimes v$ şeklinde gösterilir ve buna u ile v nin tensör çarpımı denir. Buna göre

$$u \otimes v = [(u,v)] = (u,v) + \mathcal{R}$$

dir ve $\otimes : F(U,V) \to U \otimes V$ dönüşümü bi-lineerdir.

$$(u_1 + u_2) \otimes v = (u_1 \otimes v) + (u_2 \otimes v)$$

olduğunu görelim.

$$(u_1 + u_2, v) - (u_1, v) - (u_2, v) \in \mathcal{R} \text{ ve } (cu, v) - c(u,v) \in \mathcal{R}$$

olduğundan

$$\begin{aligned} &[(u_1 + u_2, v) - (u_1, v) - (u_2, v)] = 0 \\ &[(u_1 + u_2, v)] - [(u_1, v)] - [(u_2, v)] = 0 \\ &[(u_1 + u_2, v)] = [(u_1, v)] + [(u_2, v)] \\ &(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v \end{aligned}$$

$$\begin{aligned} &[(cu, v) - c(u,v)] = 0 \\ &[(cu, v)] = [c(u,v)] = c[(u,v)] \\ &(cu) \otimes v = c(u \otimes v) \end{aligned}$$

elde edilir. Benzer yolla

$$u \otimes (v_1 + v_2) = (u \otimes v_1) + (u \otimes v_2) \text{ ve } u \otimes (cv) = c\,(u \otimes v)$$

eşitliklerinin de doğru olduğu gösterilir.

$U \otimes V$ uzayı evrensel özellik olarak bilinen aşağıdaki özelliğe sahiptir:

E herhangi bir reel vektör uzayı ve $\varphi : U \times V \longrightarrow E$ bi-lineer dönüşüm olsun. Bu takdirde $\widetilde{\varphi} : U \otimes V \longrightarrow E$ tek bir lineer dönüşümü aşağıdaki diyagramı değişmeli yapacak şekilde vardır(yani $\tilde{\varphi} \circ \otimes = \varphi$).

$$\begin{array}{ccc} U \times V & \xrightarrow{\otimes} & U \otimes V \\ & \varphi \searrow & \downarrow \tilde{\varphi} \\ & & E \end{array}$$

Şimdi böyle bir $\tilde{\varphi}$ dönüşümünün var ve tek olduğunu gösterelim. $W = F(U, V)$ nin elemanları $U \times V$ nin elemanları tarafından üretildiğinden φ yi $\bar{\varphi} : W \longrightarrow E$ tanımlı ve $\forall (u, v) \in U \times V$ için $\bar{\varphi}\,(u, v) = \varphi\,(u, v)$ koşulunu sağlayan bir lineer dönüşüme genişletebiliriz. φ bi-lineer olduğundan $\bar{\varphi}\ |_{\mathcal{R}} \equiv 0$ dır. Bunu gösterelim.

$$\begin{aligned} \bar{\varphi}\,((au + bv, w) - a\,(u, w) - b\,(v, w)) &= \varphi\,((au + bv, w) - a\,(u, w) - b\,(v, w)) \\ &= \varphi\,(au + bv, w) - a\varphi\,(u, w) - b\varphi\,(v, w) \\ &= 0 \end{aligned}$$

Aynı şekilde $\bar{\varphi}\,((u, av + bw) - a\,(u, v) - b\,(u, w)) = 0$ olduğu elde edilir. O halde

$$\begin{array}{llll} \tilde{\varphi} : & F\,(U, V)\,/R & \rightarrow & E \\ & [(u, v)] & \mapsto & \bar{\varphi}\,[(u, v)] = \bar{\varphi}\,(u, v) \end{array}$$

olur.

Şimdi de $\tilde{\varphi}$ nın iyi tanımlı olup olmadığına bakalım. $[(u, v)] = [(u', v')]$ olsun. O zaman $(u, v) - (u', v') \in \mathcal{R}$ dir.

$$\tilde{\varphi}\,[(u, v)] = \bar{\varphi}\,(u, v) \text{ ve } \tilde{\varphi}\,[(u', v')] = \bar{\varphi}\,(u', v')$$

olduğundan

$$\begin{aligned}
\tilde{\varphi}\left[(u,v)\right]-\tilde{\varphi}\left[(u',v')\right] &= \bar{\varphi}\,(u,v)-\bar{\varphi}\,(u',v') \\
&= \bar{\varphi}\,((u,v)-(u',v')) \quad (\bar{\varphi} \text{ lineer olduğundan}) \\
&= \bar{\varphi}\mid_{\mathcal{R}}((u,v)-(u',v')\in\mathcal{R}) \\
&= 0
\end{aligned}$$

elde edilir. Bu da $\tilde{\varphi}\left[(u,v)\right]=\tilde{\varphi}\left[(u',v')\right]$ olduğunu gösterir. O halde

$$\tilde{\varphi}:(F\,(U,V)\,/R)=U\otimes V\to E$$

lineer dönüşümü iyi tanımlıdır öyle ki,

$$\begin{array}{ccc}
U\times V & \xrightarrow{\otimes} & U\otimes V \\
 & \varphi\searrow & \downarrow\tilde{\varphi} \\
 & & E
\end{array}$$

diagramı değişmelidir, yani $\tilde{\varphi}\circ\otimes=\varphi$'dir.

$U\otimes V$, $u\otimes v$ formundaki elemanlar tarafından gerildiği için $\tilde{\varphi}$ tektir.

Teorem 2.1 *U ve V vektör uzayları verilsin. $\otimes:U\times V\to U\otimes V$ ve $\tilde{\otimes}:U\times V\to U\tilde{\otimes}V$ iki tensör çarpımı ise $\theta:U\otimes V\to U\tilde{\otimes}V$ izomorfizmi tek olarak vardır öyle ki,*

$$\begin{array}{ccc}
U\times V & \xrightarrow{\otimes} & U\otimes V \\
 & \tilde{\otimes}\searrow & \downarrow\theta \\
 & & U\tilde{\otimes}V
\end{array}$$

bu diagram değişmelidir. Yani, $\theta\circ\otimes=\tilde{\otimes}$ dir.

Kanıt. $\otimes$ ve $\tilde{\otimes}$ iki tensör çarpımı ise aşağıdaki diagramları değişmeli yapan tek bir θ_1 ve θ_2 lineer dönüşümleri vardır.

$$\begin{array}{ccc}
U\times V & \xrightarrow{\otimes} & U\otimes V \\
 & \tilde{\otimes}\searrow & \downarrow\theta_1 \\
 & & U\tilde{\otimes}V
\end{array}
\qquad
\begin{array}{ccc}
U\times V & \xrightarrow{\tilde{\otimes}} & U\tilde{\otimes}V \\
 & \otimes\searrow & \downarrow\theta_2 \\
 & & U\otimes V
\end{array}$$

$$\theta_1\circ\otimes=\tilde{\otimes} \qquad\qquad \theta_2\circ\tilde{\otimes}=\otimes$$

$$\begin{array}{ccc} U\times V & \stackrel{\otimes}{\longrightarrow} & U\otimes V \\ & \otimes\searrow & \downarrow I \\ & & U\otimes V \end{array} \qquad \begin{array}{ccc} U\times V & \stackrel{\otimes}{\longrightarrow} & U\tilde{\otimes}V \\ & \otimes\searrow & \downarrow \theta_2\circ\theta_1 \\ & & U\tilde{\otimes}V \end{array}$$

$$I\circ\otimes=\otimes \qquad \theta_2\circ\theta_1\circ\otimes=\theta_2\circ(\theta_1\circ\otimes)=\theta_2\circ\tilde{\otimes}=\otimes$$

Tensör çarpımının evrensel özelliğinden dolayı tek bir lineer dönüşüm olmalıydı. O halde $\theta_2\circ\theta_1=I$ olmak zorundadır. Aynı şekilde $\theta_1\circ\theta_2=I$ dır. Bu nedenle θ_1 ve θ_2 den biri diğerinin tersidir. Dolayısıyla tek bir θ_1 izomorfizmi vardır. ∎

$E\otimes F\cong F\otimes E$ olduğunu gösterelim. Bunun için $\varphi: E\times F\to F\otimes E$ ve $\psi: F\times E\to E\otimes F$ bi-lineer dönüşümlerini $\varphi(x,y)=y\otimes x$ ve $\psi(y,x)=x\otimes y$ şeklinde tanımlayalım. Evrensel özellikten

$$\begin{array}{ccc} E\times F & \stackrel{\otimes}{\longrightarrow} & E\otimes F \\ & \varphi\searrow & \downarrow f \\ & & F\otimes E \end{array}$$

tek bir $f: E\otimes F\to F\otimes E$ lineer dönüşümü vardır öyle ki,

$$f\circ\otimes(x,y)=\varphi(x,y)\quad,\quad f(\otimes(x,y))=\varphi(x,y)\quad,\quad f(x\otimes y)=y\otimes x$$

dir.

$$\begin{array}{ccc} F\times E & \stackrel{\otimes}{\longrightarrow} & F\otimes E \\ & \psi\searrow & \downarrow g \\ & & E\otimes F \end{array}$$

Evrensel özellikten tek bir $g: F\otimes E\to E\otimes F$ lineer dönüşümü vardır öyle ki;

$$g\circ\otimes(y,x)=\psi(y,x)\quad,\quad g(\otimes(y,x))=\varphi(y,x)\quad,\quad g(y\otimes x)=x\otimes y$$

dir. $(g\circ f)(x\otimes y)=g(f(x\otimes y))=g(y\otimes x)=x\otimes y$ olduğundan $g\circ f=I$ dır. Aynı şekilde $f\circ g=I$ olduğundan f ve g biri diğerinin tersi olan izomorfizmlerdir.

$V_1\otimes V_2\otimes V_3\cong(V_1\otimes V_2)\otimes V_3\cong V_1\otimes(V_2\otimes V_3)$ olduğunu gösterelim.

$$\begin{array}{ccccc} (V_1\times V_2)\times V_3 & \stackrel{\otimes\times I_{v_3}}{\longrightarrow} & (V_1\otimes V_2)\times V_3 & \stackrel{\otimes}{\longrightarrow} & (V_1\otimes V_2)\otimes V_3 \\ V_1\times(V_2\times V_3) & \stackrel{I_{v_1}\times\otimes}{\longrightarrow} & V_1\times(V_2\otimes V_3) & \stackrel{\otimes}{\longrightarrow} & V_1\otimes(V_2\otimes V_3) \end{array}$$

$$\begin{array}{ccc} V_1 \times (V_2 \times V_3) \leftrightarrow V_1 \times V_2 \times V_3 & \stackrel{\otimes_3}{\longrightarrow} & V_1 \otimes V_2 \otimes V_3 \\ & \otimes_2 \searrow & \downarrow \theta \\ & & V_1 \otimes (V_2 \otimes V_3) \end{array}$$

Evrensel özellikten dolayı tek bir θ lineer dönüşümü vardır öyle ki $\theta \circ \otimes_3 = \otimes_2$ dir.

$$\begin{array}{ccc} V_1 \times V_2 \times V_3 \leftrightarrow V_1 \times (V_2 \times V_3) & \stackrel{\otimes_2}{\longrightarrow} & V_1 \otimes (V_2 \otimes V_3) \\ & \otimes_3 \searrow & \downarrow \psi \\ & & V_1 \otimes V_2 \otimes V_3 \end{array}$$

Yine evrensel özellikten dolayı tek bir ψ lineer dönüşümü vardır öyle ki $\psi \circ \otimes_2 = \otimes_3$ dir.

$$\begin{array}{ccc} V_1 \times V_2 \times V_3 & \stackrel{\otimes_3}{\longrightarrow} & V_1 \otimes V_2 \otimes V_3 \\ & \otimes_3 \searrow & \downarrow I \\ & & V_1 \otimes V_2 \otimes V_3 \end{array} \qquad \begin{array}{ccc} V_1 \times V_2 \times V_3 & \stackrel{\otimes_3}{\longrightarrow} & V_1 \otimes V_2 \otimes V_3 \\ & \otimes_3 \searrow & \downarrow \psi \circ \theta \\ & & V_1 \otimes V_2 \otimes V_3 \end{array}$$

$$(\psi \circ \theta) \circ \otimes_3 = \psi\,(\theta \circ \otimes_3) = (\psi \circ \otimes_2) = \otimes_3$$

elde edilir. $(\psi \circ \theta) \circ \otimes_3 = \otimes_3$ olduğundan yukarıdaki diagram değişmelidir. Evrensel özellikten bu iki diagramı değişmeli yapan tek bir lineer dönüşüm olmalıydı. Bu nedenle $\psi \circ \theta = I$ olmalıdır. Aynı şekilde $\theta \circ \psi = I$ olduğu elde edilir. Dolayısıyla biri diğerinin tersi olmaktadır. Yani

$$V_1 \otimes V_2 \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$

elde edilir. Diğer izomorfizlerde benzer şekilde gösterilir.

Benzer yolla $V_1, V_2, \ldots, V_k$ gibi $k-$tane vektör uzayının da tensör çarpımı tanımlanabilir ve $V_1 \otimes \cdots \otimes V_k$ şeklinde gösterilir.

Tanım 2.1 *$\alpha \in V_1 \otimes \cdots \otimes V_k$ elemanı, $1 \leq i \leq k$ için $v_i \in V_i$ olmak üzere $\alpha = v_1 \otimes \cdots \otimes v_k$ şeklinde yazılabiliyorsa α ye ayrıştırılabilirdir denir.*

Yardımcı Teorem 2.1 *V, W iki vektör uzayı ve $A = \{e_1, e_2, \cdots, e_n\}$, $B = \{f_1, f_2, \cdots, f_m\}$ kümeleri de sırasıyla bu uzayların tabanı olsun. Bu takdirde $C = \{e_i \otimes f_j \mid i = 1, \cdots, n \text{ ve } j = 1, \cdots, m\}$ kümesi $V \otimes W$ için bir tabandır.*

Kanıt. Keyfi bir $\alpha \in V \otimes W$ alalım. Bu durumda $\alpha = \sum\limits_i v_i \otimes w_i$ şeklinde yazabiliriz. $v_i = \sum\limits_j c_{ij} e_j$ ve $w_i = \sum\limits_k d_{ik} f_k$ ise $\alpha = \sum\limits_{i,j,k} b_{ijk} e_j \otimes f_k$ olacağından C kümesi $V \otimes W$ yı gerer. Şimdi de $\sum\limits_{i,j,k} b_{ijk} e_j \otimes f_k = 0$ ise $b_{ijk} = 0$ olduğunu gösterelim. Tensör çarpımının tanımından $\sum\limits_{i,j,k} b_{ijk}(e_j, f_k) \in \mathcal{R}$ olacaktır. Ancak $\sum\limits_{i,j,k} b_{ijk}(e_j, f_k)$ toplamı, $\mathcal{R}$ deki elemanlar tarafından üretilemeyeceğinden $b_{ijk} = 0$ olmalıdır. O halde C kümesi $V \otimes W$ için bir tabandır. ■

Sonuç 2.1 *$V_1, V_2, \ldots, V_k$ vektör uzaylarının tabanları sırasıyla $B_1, B_2, \ldots, B_k$ ise $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ vektör uzayının tabanı*

$$B = \{v_1 \otimes v_2 \otimes \cdots \otimes v_k \mid 1 \leq i \leq k \text{ için } v_i \in B_i\}$$

dır.

E, F, G, H vektör uzayları, $\varphi : E \rightarrow F$ ve $\psi : G \rightarrow H$ lineer dönüşümler olsun. Bu durumda

$$\lambda : E \times G \rightarrow F \otimes H$$
$$\lambda(x, y) = \varphi(x) \otimes \psi(y)$$

bi-lineer dönüşümünü tanımlayalım. Bi-lineer olduğu kolaylıkla doğrulanabilir.

$$\begin{array}{ccc} E \times G & \xrightarrow{\otimes} & E \otimes G \\ & \lambda \searrow & \downarrow \chi \\ & & F \otimes H \end{array}$$

Evrensel özellikten dolayı $\chi : E \otimes G \longrightarrow F \otimes H$ tek bir lineer dönüşümü vardır öyle ki $\chi(x \otimes y) = \varphi(x) \otimes \psi(y)$ dir. Bu şekilde tanımlı χ dönüşümüne φ ve ψ dönüşümlerinin tensör (ya da Kronecker) çarpımı denir. Bazen $\chi = \varphi \otimes \psi$ gösterimi de kullanılır.

Yardımcı Teorem 2.2 *E, F, G, H vektör uzayları, $\varphi : E \longrightarrow F$ ve $\psi : G \longrightarrow H$ lineer dönüşümler ayrıca bu uzayların sırasıyla $\{e_1, e_2, \cdots, e_m\}$, $\{f_1, f_2, \cdots, f_n\}$,$\{g_1, g_2, \cdots, g_p\}$ ve $\{h_1, h_2, \cdots, h_q\}$ tabanları verilsin. φ ve ψ nin bu tabanlara göre matrisleri de sırasıyla $A = (a_{ij})$ ve $B = (b_{ij})$ olsun. Bu durumda $\chi = \varphi \otimes \psi$ dönüşümünün*

$$\{e_i \otimes g_j : i = 1, ..., m, j = 1, ..., p\} \text{ ve } \{f_i \otimes h_j : i = 1, ..., n, j = 1, ..., q\}$$

tabanlarına göre matrisi $nq \times mp$ tipindeki $C = (a_{ij}B)$ matrisidir(Bu matrise de A ve B matrislerinin Kronecker çarpımı denir.).

Kanıt. Şimdi χ dönüşümünün $\{e_i \otimes g_j : i = 1, ..., m,\ j = 1, ..., p\}$ ve $\{f_i \otimes h_j : i = 1, ..., n,\ j = 1, ..., q\}$ tabanlarına göre matrisini bulalım.

$$\chi(e_i \otimes g_j) = \varphi(e_i) \otimes \psi(g_j)$$

olduğundan,

$$\begin{aligned} \chi(e_i \otimes g_j) &= (a_{1i}f_1 + a_{2i}f_2 + \cdots + a_{ni}f_n) \otimes (b_{1j}h_1 + b_{2j}h_2 + \cdots + b_{qj}h_q) \\ &= (a_{1i}b_{1j})\, f_1 \otimes h_1 + \cdots + (a_{1i}b_{qj})\, f_1 \otimes h_q + \cdots + \\ &\quad (a_{ni}b_{1j})\, f_n \otimes h_1 + \cdots + (a_{ni}b_{qj})\, f_n \otimes h_q \end{aligned}$$

bulunur. O halde χ dönüşümünün verilen tabanlara göre matrisini açık olarak yazarsak aşağıdaki matris elde edilir.

$$C = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1p} & \cdots & a_{1m}b_{11} & \cdots & a_{1m}b_{1p} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{11}b_{q1} & a_{11}b_{q2} & \cdots & a_{11}b_{qp} & \cdots & a_{1m}b_{q1} & \cdots & a_{1m}b_{qp} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}b_{11} & a_{n1}b_{12} & \cdots & a_{n1}b_{1p} & \cdots & a_{nm}b_{11} & \cdots & a_{nm}b_{1p} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1}b_{q1} & a_{n1}b_{q2} & \cdots & a_{n1}b_{qp} & \cdots & a_{nm}b_{q1} & \cdots & a_{nm}b_{qp} \end{pmatrix}$$

Elde ettiğimiz bu C matrisi $C = (a_{ij}B)_{nq \times mp}$ özelliğindedir. ■

Yukarıda belirtilen $(\varphi, \psi) \rightarrow \chi$ eşlemesi

$$\beta : L(E; F) \times L(G; H) \rightarrow L(E \otimes G; F \otimes H)$$

şeklinde bir bi-lineer dönüşüm tanımlar. Yine tensör çarpımının evrensel özelliğinden

$$f : L(E; F) \otimes L(G; H) \rightarrow L(E \otimes G; F \otimes H)$$

tanımlı ve tek türlü belirli f lineer dönüşümü her $(\varphi, \psi) \in L(E; F) \times L(G; H)$ için $\beta(\varphi, \psi) = f(\varphi \otimes \psi)$ koşulunu sağlayacak şekilde vardır.

Önerme 2.1 *$L(E; F)$ ve $L(G; H)$ vektör uzaylarının tensör çarpımı $L(E; F) \otimes L(G; H)$ olsun.*

$$\beta : L(E; F) \times L(G; H) \rightarrow L(E \otimes G; F \otimes H)$$

bi-lineer dönüşümünün belirlediği

$$f : L(E; F) \otimes L(G; H) \rightarrow L(E \otimes G; F \otimes H)$$

lineer dönüşümü birebirdir.

Kanıt. $w \in L(E; F) \otimes L(G; H)$ elemanını alalım, öyle ki $f(w) = 0$ olsun. $w \neq 0$ olduğunu kabul edelim. O zaman $\varphi_i \in L(E; F)$, $\psi_i \in L(G; H)$ olmak üzere,

$$w = \sum_{i=1}^{r} \varphi_i \otimes \psi_i$$

olur. Burada φ_i ve ψ_i ler lineer bağımsızdır. $f(w) = \sum_{i=1}^{r} \beta(\varphi_i, \psi_i)$ dir. $f(w) = 0$ olduğundan $\forall x \in E, y \in G$ çifti için

$$\sum_{i=1}^{r} \varphi_i(x) \otimes \psi_i(y) = 0 \qquad (2.1)$$

olur. Şimdi $a \in E$ vektörünü seçelim öyle ki $\varphi_1(a) \neq 0$ olsun. $\{\varphi_1(a), \varphi_2(a), \ldots, \varphi_r(a)\}$ kümesinde lineer bağımsız vektörlerin maksimal sayısı $p \geq 1$ olsun.

Yani her $2 \leq i \leq r$ için $\varphi_i(a) \neq 0$ ve $\{\varphi_1(a), \varphi_2(a), \ldots, \varphi_i(a)\}$ kümesi ile lineer bağımsız olacak şekilde hem sıfırdan farklı hem de lineer bağımsız bir $\{\varphi_{i_1}(a), \ldots, \varphi_{i_k}(a)\}$ kümesini elde edelim. İndislemede tekrar düzenleme yaparsak $\{\varphi_1(a), \ldots, \varphi_p(a)\}$ olur. O halde $j = p+1, \ldots, r$ için

$$\varphi_j(a) = \sum_{i=1}^{p} \lambda_{ji} \varphi_i(a)$$

olur. $y \in F$ için (2.1) eşitliğinden,

$$\sum_{i=1}^{p} \varphi_i(a) \otimes \psi_i(y) + \sum_{j=p+1}^{r} \left(\sum_{i=1}^{p} \lambda_{ji} \varphi_i(a) \right) \otimes \psi_j(y) = 0$$

$$\sum_{i=1}^{p} \varphi_i(a) \otimes \left(\sum_{j=p+1}^{r} \lambda_{ji} \psi_j(y) + \psi_i(y) \right) = 0$$

sonuçları elde edilir. $i = 1, \ldots, p$ ve $\forall y \in F$ için $\varphi_i(a)$ lar lineer bağımsız olduğundan

$$\psi_i(y) + \sum_{j=p+1}^{r} \lambda_{ji} \psi_j(y) = 0$$

$$\psi_i + \sum_{j=p+1}^{r} \lambda_{ji} \psi_j = 0$$

$$\psi_i = -\sum_{j=p+1}^{r} \lambda_{ji} \psi_j$$

sonuçları bulunur. Başlangıçta ψ_j dönüşümlerini lineer bağımsız olarak almıştık. Oysa yukarıdaki sonuç hipotezimizle çelişti. O halde f birebirdir. ∎

Vektör uzaylarında olduğu gibi iki cebirin de tensör çarpımı tanımlanabilir. U ve V reel vektör uzayları cebir yapısına sahip olsunlar, bu durumda $U \otimes V$ nin vektör uzayı olduğunu biliyoruz. $U \otimes V$ üzerinde vektörlerin çarpması da üreteçler üzerinden $u_1, u_2 \in U$ ve $v_1, v_2 \in V$ olmak üzere

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (u_1 u_2) \otimes (v_1 v_2)$$

şeklinde tanımlanır (Bu tanımlama tensör çarpımın evrensel özelliğinden elde edilir. Bakınız [7]).

Tanım 2.2 *V vektör uzayı olsun. Her $n \geq 0$ tamsayısı için V nin n. tensör kuvveti,*

$$T^n(V) = \begin{cases} \mathbb{R} & , \; n = 0 \text{ ise} \\ V & , \; n = 1 \text{ ise} \\ V \otimes \cdots \otimes V & , \; n \geq 2 \text{ ise} \end{cases}$$

olarak tanımlanır ve

$$\begin{aligned} T^0(V) \otimes T^n(V) &= T^n(V) \otimes T^0(V) = T^n(V) \\ T^n(V) \otimes T^m(V) &= T^{n+m}(V) \end{aligned}$$

özellikleri vardır.

$T(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots$ direkt toplamını düşünelim. $T(V)$ nin elemanları, $p = 0, 1, 2, \ldots$ için $z_p \in T^p(V)$ olmak üzere $(z_0, z_1, \ldots)$ dizisidir öyle ki her bir dizide z_p nin sadece sonlu tanesi sıfırdan farklıdır. Şimdi

$$\begin{aligned} T(V) \times T(V) &\rightarrow T(V) \\ (u, v) &\longmapsto u \cdot v \end{aligned}$$

bi-lineer dönüşümünü, $u = \sum_p u_p$ ve $v = \sum_q v_q$ için $u \cdot v = \sum_{p,q} u_p \otimes v_q$ olacak şekilde tanımlayalım. Bu çarpım $T(V)$ yi birleşmeli yapar. $(1, 0, 0, \ldots)$ dizisi birim elemanıdır. $T(V)$ ye V vektör uzayı üzerinde bir tensör cebri denir. 1 skaleri ve V nin elemanları tensör cebirini üretir. Şimdi tensör cebirinin evrensel özelliğini tanımlayalım. A birleşmeli ve birimli keyfi bir cebir olsun. $i : V \rightarrow T(V)$ gömme dönüşümü ve $u : V \rightarrow A$ lineer dönüşüm ise, tek bir $U : T(V) \rightarrow A$ cebir homomorfizmi aşağıdaki diagramı değişmeli yapacak şekilde (yani $U \circ i = u$) vardır.

$$\begin{array}{ccc} V & \xrightarrow{i} & T(V) \\ & u \searrow & \downarrow U \\ & & A \end{array}$$

Daha detaylı bilgi için [7] ye bakınız.

3 CLİFFORD CEBİRİ

V sonlu boyutlu bir reel vektör uzayı ve Q da V üzerinde bir kuadratik form iken (V,Q) ikilisine bir kuadratik uzay demiştik. Bu bölümde (V,Q) ikilisinin belirlediği V yi (aslında bir kopyasını) içeren birimli, birleşmeli reel bir cebir tanımlayacağız.

Tanım 3.1 *(V,Q) bir reel kuadratik uzay, $T(V) = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \ldots$ V vektör uzayının tensör cebri ve $I(Q)$ da bu cebir içerisinde $\{v \otimes v - Q(v) \cdot 1\}$ formundaki elemanlar tarafından üretilen iki taraflı ideal olsun. Bu durumda $Cl(V,Q) = T(V)/I(Q)$ bölüm cebirine (V,Q) nun Clifford cebiri denir.*

$Cl(V,Q)$ Clifford cebiri aşağıdaki özelliklere sahiptir:

1. $Cl(V,Q)$ birimli ve birleşmeli bir cebirdir. Tensör cebri birimli olduğundan tensör cebrinin biriminin bölüm dönüşümü (doğal izdüşüm)

$$\begin{array}{ccc} T(V) & \xrightarrow{\pi} & T(V)/I(Q) = Cl(V,Q) \\ v & \mapsto & [v] = v + I(Q) \end{array}$$

   altındaki görüntüsü $Cl(V,Q)$ Clifford cebirinin birimidir ($[1] \cdot [\alpha] = [1 \otimes \alpha] = [\alpha]$).

   Şimdi $Cl(V,Q)$ Clifford cebirinin birleşme özelliğini tensör çarpımının birleşme özelliğini kullanarak gösterelim.

   $[\alpha], [\beta], [\gamma] \in Cl(V,Q)$, ($\alpha, \beta, \gamma \in T(V)$) olsun.

$$\begin{aligned} [\alpha] \cdot ([\beta] \cdot [\gamma]) &= [\alpha] \cdot ([\beta \otimes \gamma]) \\ &= [\alpha \otimes (\beta \otimes \gamma)] \\ &= [(\alpha \otimes \beta) \otimes \gamma] \\ &= [\alpha \otimes \beta] \cdot [\gamma] \\ &= ([\alpha] \cdot [\beta]) \cdot [\gamma] \end{aligned}$$

   olduğundan $Cl(V,Q)$ Clifford cebiri birimli ve birleşmeli bir cebirdir.

2. $V \xrightarrow{j=\pi \circ i} T(V)/I(Q) = Cl(V,Q)$ dönüşümü altında V nin görüntüsü, V nin $Cl(V,Q)$ içindeki bir kopyasıdır, üstelik her $v \in V$ için $j(v)^2 = Q(v) \cdot 1$ olur.

Her $v \in V$ için,

$$\begin{array}{ccccc} V & \xrightarrow{i} & T(V) & \xrightarrow{\pi} & T(V)/I(Q) = Cl(V,Q) \\ v & \mapsto & v & \mapsto & [v] = v + I(Q) \end{array}$$

$j(v) = \pi \circ i(v) = \pi(v) = [v]$ olduğundan

$$j(v)^2 = j(v) \cdot j(v) = [v] \cdot [v] = [v \otimes v]$$

olur. Burada $[v \otimes v] = [t]$ diyecek olursak, $v \otimes v - t \in I(Q)$ olacağından $t = Q(v) \cdot 1$ olmak zorundadır. Bu durumda

$$j(v)^2 = [v \otimes v] = [Q(v) \cdot 1] = Q(v) \cdot [1] = Q(v) \cdot 1$$

elde edilir.

3. $Cl(V,Q)$, 1 ve $j(V) = V$ ler tarafından üretilir. $j : V \to Cl(V,Q)$ lineer dönüşümü birebirdir. $j(V) \subset Cl(V,Q)$ kümesi cebiri çarpımsal olarak üretir.

4. $Cl(V,Q)$ **Clifford cebirinin evrensel özelliği:** A, $\mathbb{R}$ üzerinde birimli ve birleşmeli bir cebir ve $u : V \to A$ lineer dönüşümü her $v \in V$ için $u(v)^2 = Q(v) \cdot 1$ koşulunu sağlıyorsa, bu takdirde

$$\begin{array}{ccc} V & \xrightarrow{j} & Cl(V,Q) \\ & u \searrow & \downarrow \tilde{U} \\ & & A \end{array}$$

diyagramı değişmeli olacak şekilde (yani $\tilde{U} \circ j = u$) tek bir

$$\tilde{U} : Cl(V,Q) \to A$$

cebir homomorfizmi vardır.

Şimdi böyle bir $\tilde{U}$ nın var ve tek olduğunu gösterelim. $u : V \longrightarrow A$ lineer dönüşüm ve $u(v)^2 = Q(v) \cdot 1$ olsun.

u lineer dönüşüm olduğundan tensör cebrinin evrensel özelliğinden U : $T(V) \to A$ tek bir cebir homomorfizmine genişletilebilir öyle ki,

$$\begin{array}{ccc} V & \xrightarrow{i} & T(V) \\ & u\searrow & \downarrow U \\ & & A \end{array}$$

diagramı değişmeli olur, yani $U \circ i = u$ dur.

$U(v_1 \otimes \ldots \otimes v_k) = u(v_1)\ldots u(v_k)$ dır ve $I(Q) \subset \ker U$ dur. Gerçekten $\alpha \otimes (v \otimes v - Q(v)\cdot 1) \otimes \beta \in I(Q)$ için,

$$\begin{aligned} U(\alpha \otimes (v \otimes v - Q(v)\cdot 1) \otimes \beta) &= u(\alpha)\,[u(v)\,u(v) - Q(v)\cdot 1]\,u(\beta) \\ &= u(\alpha).0.u(\beta) = 0 \end{aligned}$$

elde edilir. O halde $I(Q) \subset \ker U$ dir.

$$\begin{array}{rclcl} \tilde{U}: T(V)/I(Q) & = & Cl(V,Q) & \longrightarrow & A \\ & & [\alpha] & \longmapsto & \tilde{U}([\alpha]) = U(\alpha) \end{array}$$

şeklinde tanımlanan $\tilde{U}$ nın iyi tanımlı olduğunu gösterelim. $[\alpha] = [\beta]$ dersek, $\alpha - \beta \in I(Q)$ olur. Bu durumda $U(\alpha - \beta) = U(\alpha) - U(\beta) = 0$ dır. Buradan $\tilde{U}([\alpha]) = \tilde{U}([\beta])$ elde edilir. Her $v \in V$ için $j(v) \in Cl(V,Q)$ olduğundan $\left(\tilde{U} \circ j\right)(v) = \tilde{U}(j(v)) = u(v)$ dir.

Şimdi de $\tilde{U}$ nın cebir homomorfizmi olduğunu gösterelim. U dönüşümünün cebir homomorfizmi olduğunu biliyoruz. $[\alpha], [\beta] \in Cl(V,Q)$ ve $\lambda \in \mathbb{R}$ için,

$$\begin{aligned} &\tilde{U}([\alpha].[\beta]) = \tilde{U}([\alpha \otimes \beta]) = U(\alpha \otimes \beta) = U(\alpha).U(\beta) = \tilde{U}([\alpha]).\tilde{U}([\beta]) \\ &\tilde{U}([\alpha] + [\beta]) = \tilde{U}([\alpha + \beta]) = U(\alpha + \beta) = U(\alpha) + U(\beta) = \tilde{U}([\alpha]) + \tilde{U}([\beta]) \\ &\tilde{U}(\lambda.[\alpha]) = \tilde{U}([\lambda \otimes \alpha]) = U(\lambda \otimes \alpha) = \lambda.U(\alpha) = \lambda.\tilde{U}([\alpha]) \end{aligned}$$

koşulları sağlandığından $\tilde{U}$ bir cebir homomorfizmidir.

Önerme 3.1 *Bir quadratik formun $Cl(V,Q)$ Clifford cebirinin*

$$\beta : Cl(V,Q) \to Cl(V,Q)$$

bir involusyonu vardır öyle ki aşağıdaki koşullar sağlanır.

(i) β *bir cebir homomorfizmi ve* $\beta^2 = I$ *dır.*

(ii)

$$Cl_0(V,Q) = \{x \in Cl(V,Q) \mid \beta(x) = x\}$$

ve

$$Cl_1(V,Q) = \{x \in Cl(V,Q) \mid \beta(x) = -x\}$$

ise $Cl(V,Q) = Cl_0(V,Q) \oplus Cl_1(V,Q)$ *şeklinde yazılabilir (Bakınız [6]).*

Her clifford cebiri $\beta : Cl(V,Q) \to Cl(V,Q)$ involusyonunun yanısıra bir anti involusyon taşır. Bunu şu şekilde açıklayalım. A bir cebir ise $x * y = y.x$ çarpımı ile A kümesi üzerinde yeni bir A^* cebrini tanımlayalım. (V,Q) kuadratik formuna karşılık gelen clifford cebri $Cl(V,Q)$ olsun. $A^* = Cl(V,Q)^*$ cebrini düşünelim. $V \xrightarrow{j} A^* = Cl(V,Q)^*$ lineer dönüşümü için

$$j(v) * j(v) = j(v).j(v) = j(v)^2 = Q(v).1$$

ilişkisi A^* cebirinde de geçerlidir. O halde Clifford cebrinin evrensel özelliğinden tek bir $\gamma : Cl(V,Q) \to Cl(V,Q)^*$ cebir homomorfizmi vardır öyle ki her $v \in V$ için $j(v) = \gamma \circ j(v)$ koşulu sağlanır.

$$\begin{array}{ccc} V & \xrightarrow{j} & Cl(V,Q) \\ & j\searrow & \downarrow\gamma \\ & & Cl(V,Q)^* \end{array}$$

$\gamma : Cl(V,Q) \to Cl(V,Q)$ dönüşümünün aşağıdaki özellikleri vardır.

Önerme 3.2 *Her Clifford dönüşümü için aşağıdaki özellikleri sağlayan bir* $\gamma : Cl(V,Q) \to Cl(V,Q)$ *lineer dönüşümü vardır.*

(i) γ *lineerdir.*

(ii) $\gamma \circ \gamma = I$ *(*γ *bir involusyon) dır.*

(iii) $\forall v \in V \subset Cl(Q)$ *için* $\gamma(v) = v$ *dir.*

(iv) $x, y \in Cl(Q)$ *için* $\gamma(x.y) = \gamma(y).\gamma(x)$ *dir.*

Tanım 3.2 *A bir $\mathbb{R}$-cebri olsun. A_0 ve A_1, $k = i + j \pmod 2$ için $A_iA_j \subset A_k$ koşulunu sağlayan $\mathbb{R}$-vektör uzayları olmak üzere A cebri $A_0 \oplus A_1$ şeklinde yazılabiliyorsa A cebrine $\mathbb{Z}_2 -$ graded cebri denir. $T(V)$ tensör cebri, $\mathbb{Z}_2$-graded cebrine örnek olarak gösterilebilir. Burada*

$$\begin{aligned}(T(V))_0 &= \mathbb{R} \oplus (V \otimes V) \oplus \cdots \\ (T(V))_1 &= V \oplus (V \otimes V \otimes V) \oplus \cdots\end{aligned}$$

şeklindedir. Ayrıca $Cl(V,Q)$ Clifford cebride $\mathbb{Z}_2$-graded cebridir. $Cl(V,Q)$ Clifford cebri için $Cl_0(V,Q)$, V vektör uzayının elemanlarının çift sayıdakilerinin, $Cl_1(V,Q)$, V vektör uzayının elemanlarının tek sayıdakilerinin çarpımı ile üretilir.

Bir ortogonal toplamın Clifford cebrini bulabiliriz. Bunun için $A = A_0 \oplus A_1$ ve $B = B_0 \oplus B_1$ olmak üzere $\mathbb{Z}_2$-graded $\mathbb{R}$-cebirlerinin $A\hat{\otimes}B$ graded tensör çarpımına ihtiyacımız olacak. Bu çarpım $a, a' \in A$, $b, b' \in B$ için,

$$(a \otimes b)(a' \otimes b') = (-1)^{\partial(b)\partial(a')} aa' \otimes bb'$$

şeklinde tanımlanır. Burada $\partial(b)$ b'nin derecesi $\partial(a')$ ise a' nin derecesidir(bu derece 0 ya da 1 olacaktır.). $A\hat{\otimes}B$ cebri,

$$\begin{aligned}\left(A\hat{\otimes}B\right)_0 &= A_0 \otimes B_0 + A_1 \otimes B_1 \\ \left(A\hat{\otimes}B\right)_1 &= A_1 \otimes B_0 + A_0 \otimes B_1\end{aligned}$$

ile $\mathbb{Z}_2$-graded cebridir. (Bakınız [9])

Önceki bölümde (V_1, Q_1) ve (V_2, Q_2) şeklinde iki kuadratik uzay verildiğinde, $V = V_1 \oplus V_2$ vektör uzayı üzerinde de Q_1, Q_2 yardımıyla bir kuadratik form tanımlanabildiğine işaret etmiştik ve bunu $Q = Q_1 \oplus Q_2$ şeklinde göstermiştik. Şimdi bu kuadratik uzaylara karşılık gelen Clifford cebirleri arasındaki ilişkiyi veren bir teoremi ifade edelim. Kısalık için $Cl(V,Q)$ yerine $Cl(Q)$ gösterimini kullanacağız.

Önerme 3.3 *$Cl(Q_1 \oplus Q_2) \cong Cl(Q_1)\hat{\otimes}Cl(Q_2)$ dir.*

Kanıt.

$$u : V_1 \oplus V_2 \longrightarrow Cl(Q_1)\hat{\otimes}Cl(Q_2)$$

$$u(v_1 + v_2) = j_1(v_1) \otimes 1 + 1 \otimes j_2(v_2)$$

şeklinde u lineer dönüşümünü tanımlayalım. Burada j_1 *ve* j_2

$$\begin{array}{llllll} j_1: & V_1 & \to & Cl(Q_1) & j_1(v_1)^2 & = Q_1(v_1)\cdot 1 \\ j_2: & V_2 & \to & Cl(Q_2) & j_2(v_2)^2 & = Q_2(v_2)\cdot 1 \end{array}$$

koşullarını sağlayan lineer dönüşümlerdir.

$$\begin{aligned} u(v_1+v_2)^2 &= (j_1(v_1)\otimes 1 + 1\otimes j_2(v_2))(j_1(v_1)\otimes 1 + 1\otimes j_2(v_2)) \\ &= j_1(v_1)^2\otimes 1 + j_1(v_1)\otimes j_2(v_2) - j_1(v_1)\otimes j_2(v_2) + 1\otimes j_2(v_2)^2 \\ &= Q_1(v_1)\cdot 1\otimes 1 + 1\otimes Q_2(v_2)\cdot 1 \\ &= (Q_1(v_1)+Q_2(v_2))\cdot(1\otimes 1) \\ &= (Q_1\oplus Q_2)(v_1+v_2)\cdot 1 = Q(v_1+v_2)\cdot 1 \end{aligned}$$

$u(v_1+v_2)^2 = Q(v_1+v_2)\cdot 1$ *olduğundan Clifford cebirinin evrensel özelliğinden dolayı, tek bir* $g: Cl(Q_1\oplus Q_2)\to Cl(Q_1)\hat{\otimes}Cl(Q_2)$ *cebir homomorfizmi vardır öyle ki* $g\circ j = u$ *dur.*

$$\begin{array}{ccc} V_1\oplus V_2 & \xrightarrow{j} & Cl(Q_1\oplus Q_2) \\ & u\searrow & \downarrow g \\ & & Cl(Q_1)\hat{\otimes}Cl(Q_2) \end{array}$$

Şimdi

$$\begin{array}{cccccc} V_1 & \xhookrightarrow{i_1} & V_1\oplus V_2 & V_2 & \xhookrightarrow{i_2} & V_1\oplus V_2 \\ v_1 & \mapsto & (v_1,0) & v_2 & \mapsto & (0,v_2) \end{array}$$

gömme dönüşümlerini alalım. Bu dönüşümler izometridirler.

$$\begin{array}{llllll} V_1 & \xhookrightarrow{i_1} & V_1\oplus V_2 & \xrightarrow{j} & Cl(Q_1\oplus Q_2) & j\circ i_1 \text{ lineer dönüşümdür} \\ V_2 & \xhookrightarrow{i_2} & V_1\oplus V_2 & \xrightarrow{j} & Cl(Q_1\oplus Q_2) & j\circ i_2 \text{ lineer dönüşümdür.} \end{array}$$

$$\begin{array}{ccc} V_1 & \xrightarrow{j_1} & Cl(Q_1) \\ & j\circ i_1\searrow & \downarrow i_c \\ & & Cl(Q_1\oplus Q_2) \end{array} \qquad \begin{array}{ccc} V_2 & \xrightarrow{j_2} & Cl(Q_2) \\ & j\circ i_2\searrow & \downarrow j_c \\ & & Cl(Q_1\oplus Q_2) \end{array}$$

Clifford cebirinin evrensel özelliğinden tek türlü i_c *ve* j_c *cebir homomorfizmleri vardır. Buradan*

$$\begin{aligned} (j\circ i_1)^2(v_1) &= j(i_1(v_1))\,j(i_1(v_1)) = j^2((v_1,0)) \\ &= (Q_1+Q_2)(v_1,0)\cdot 1 = Q_1(v_1)\cdot 1 \end{aligned}$$

$$\begin{aligned}(j \circ i_2)^2(v_2) &= j(i_2(v_2))\, j(i_2(v_2)) = j^2((0,v_2)) \\ &= (Q_1+Q_2)(0,v_2)\cdot 1 = Q_2(v_2)\cdot 1\end{aligned}$$

sonuçları elde edilir. Şimdi $a \in Cl(Q_1)$, $b \in Cl(Q_2)$ *olmak üzere*

$$\begin{aligned}&f : Cl(Q_1)\hat{\otimes} Cl(Q_2) \to Cl(Q_1 \oplus Q_2) \\ &f(a\otimes b) = i_c(a) . j_c(b)\end{aligned}$$

şeklinde f *lineer dönüşümünü tanımlayalım.* f *nin cebir homomorfizmi olduğunu göstermek için*

$$i_c(a) . j_c(b) = (-1)^{\partial(a)\partial(b)}\, j_c(b) . i_c(a)$$

olduğunu göstermek yeterlidir. $a \in Cl(Q_1)$ *ve* $b \in Cl(Q_2)$ *olduğundan*

$$\begin{aligned}x_i &\in V_1 \text{ için } a = x_1 \ldots x_p \\ y_j &\in V_2 \text{ için } b = y_1 \ldots y_q \quad \text{dersek}\end{aligned}$$

$$\begin{aligned}i_c(a) . j_c(b) &= i_c(x_1 \ldots x_p)\, j_c(y_1 \ldots y_q) \\ &= i_c(x_1)\ldots i_c(x_p)\, j_c(y_1)\ldots j_c(y_q)\end{aligned} \tag{3.1}$$

elde edilir. Her $i_c(x_i)$ *ve* $j_c(y_j)$ *vektörleri,* $V_1 \oplus V_2$ *deki iç çarpıma karşılık ortagonal olduğundan*

$$i_c(x_i)\, j_c(y_j) = -j_c(y_j)\, i_c(x_i)$$

dir. Bu durumda (3.1) eşitliği

$$i_c(a) . j_c(b) = (-1)^{pq}\, j_c(y_1)\ldots j_c(y_q)\, i_c(x_1)\ldots i_c(x_p) = (-1)^{pq}\, j_c(b)\, i_c(a)$$

şekline dönüşür. O halde

$$f : Cl(Q_1)\hat{\otimes} Cl(Q_2) \to Cl(Q_1 \oplus Q_2)$$

lineer dönüşümü homomorfizmdir. Ayrıca

$$g : Cl(Q_1 \oplus Q_2) \to Cl(Q_1)\hat{\otimes} Cl(Q_2)$$

lineer dönüşümünün homomorfizm olduğunu göstermiştik. f *ve* g *nin tanımından*

$$\begin{aligned} g\circ f(x\otimes 1) &= g(f(x\otimes 1)) \\ &= g(i_c(x)\, j_c(1)) \\ &= g(i_c(x)) \\ &= g(x\oplus 0) \\ &= u(x\oplus 0) \quad (g \text{ homomorfizm olduğundan}) \\ &= j_1(x)\otimes 1 = x\otimes 1 \end{aligned}$$

$g\circ f(x\otimes 1) = x\otimes 1$ *elde edilir. Benzer şekilde* $g\circ f(1\otimes y) = 1\otimes y$ *bulunur. Burada* $y=1$ *alınırsa* $g\circ f(1\otimes 1) = 1\otimes 1$ *elde edilir.* $Cl(Q_1)\hat{\otimes}Cl(Q_2)$ *cebri,* $x\otimes 1, 1\otimes y$ *ve* $1\otimes 1$ *elemanları tarafından üretildiğinden* $g\circ f = I$ *sonucu elde edilir. Diğer taraftan her* $x\in V_1$ *ve* $y\in V_2$ *için*

$$\begin{aligned} f\circ g(x\oplus y) &= f(u(x\oplus y)) \\ &= f(j_1(x)\otimes 1 + 1\otimes j_2(y)) \\ &= f(j_1(x)\otimes 1) + f(1\otimes j_2(y)) \\ &= i_c(j_1(x)) + j_c(j_2(y)) \\ &= i_c(x) + j_c(y) \\ &= x\oplus y \end{aligned}$$

$f\circ g(x\oplus y) = x\oplus y$ *olur. Benzer şekilde* $f\circ g(1\oplus 1) = 1\oplus 1$ *elde edilir. O halde* $f\circ g = I$ *dır. Tüm bu eşitliklerden dolayı* f *ve* g *biri diğerinin tersi olan izomorfizmlerdir.* ■

Şimdi de Clifford cebirlerinin belirlenmesinde faydalı olan iki teorem vereceğiz.

Önerme 3.4 *V n-boyutlu bir vektör uzayı olsun. Bu durumda $Cl(Q)$ vektör uzayının boyutu 2^n dir. Yani, $dim_{\mathbb{R}}Cl(Q) = 2^n$ dir.*

Kanıt. Teorem (1.4) den bir kuadratik form, n tane bir boyutlu kuadratik formların toplamı şeklinde yazılabilir. O halde $Q = Q_1\oplus\cdots\oplus Q_n$ dir.

Bir boyutlu kuadratik formun clifford cebiri $Cl_0(Q) = \mathbb{R}$, $Cl_1(Q) = \mathbb{R}.e$ ve $e^2 = Q(e) \cdot 1$ dir. O halde bir boyutlu kuadratik form için $\dim_{\mathbb{R}} Cl(Q) = 2$ dir. Önerme (3.3) den

$$Cl(Q) = Cl(\mathbb{R}, Q_1) \hat{\otimes} \cdots \hat{\otimes} Cl(\mathbb{R}, Q_n)$$

olur. Böylece $\dim_{\mathbb{R}} Cl(Q) = 2^n$ elde edilir. ■

Önerme 3.5 *(V, b) bi-lineer form ve V nin bir tabanı $\{v_1, \ldots, v_n\}$ ise $i \neq j$ iken $b(v_i, v_j) = 0$ olsun. O zaman $Cl(Q)$ Clifford cebiri, $\{v_1, \ldots, v_n\} \in V \subset Cl(Q)$ elemanları tarafından üretilir ve $v_i^2 = Q(v_i).1$, $i \neq j$ iken $v_i v_j + v_j v_i = 0$ eşitlikleri sağlanır. $Cl(Q)$ vektör uzayının tabanı*

$$1 \text{ ve } v_{i_1} \ldots v_{i_s} \quad (1 \leq i_1 < i_2 < \cdots < i_s \leq n \text{ ve } 1 \leq s \leq n)$$

elemanları tarafından oluşturulur.

Kanıt. $V \subset Cl(Q)$, $Cl(Q)$ yu çarpımsal olarak üretir. $\{v_1, \ldots, v_n\}$, V vektör uzayının bir tabanı olsun. Böylece $\{v_1, \ldots, v_n\}$, $Cl(Q)$ cebrini de üretir. Ayrıca $v_i^2 = Q(v_i) \cdot 1$ ve

$$(v_i + v_j)^2 = Q(v_i + v_j) \cdot 1 = Q(v_i) \cdot 1 + Q(v_j) \cdot 1 = v_i^2 + v_j^2$$

dir. Buradan $i \neq j$ iken $v_i v_j + v_j v_i = 0$ sonucu elde edilir. Burada 2^n eleman $Cl(Q)$ yu üretir. Sonuç olarak $\dim Cl(Q) = 2^n$ olduğundan 2^n tane eleman bir taban olmak zorundadır. ■

Örnek 3.1 *V vektör uzayı üzerinde $Q = 0$ quadratik formunu düşünelim. Bu durumda $Cl(V, Q) = \Lambda(V)$ ya V nin dış cebri denir.*

Herhangi (V, Q) çiftine karşılık $Cl(V, Q)$ Clifford cebrini tanımladık. Acaba V üzerinde Q_1 ve Q_2 gibi denk iki kuadratik form verilirse karşılık gelen Clifford cebirleri arasındaki ilişki nedir? Aşağıdaki önerme bu ilişkiyi açıklar.

Önerme 3.6 *V üzerinde Q_1 ve Q_2 gibi denk iki kuadratik form verilsin. Bu kuadratik formlara karşılık gelen Clifford cebirleri $Cl(V, Q_1)$ ve $Cl(V, Q_2)$ ise bir $f : Cl(V, Q_1) \to Cl(V, Q_2)$ izomorfizmi vardır.*

Kanıt. (V, Q_1) *kuadratik uzayına karşılık gelen Clifford cebiri* $Cl(V, Q_1)$ *olduğundan Clifford cebrinin evrensel özelliğinden her* $v \in V$ *için* $j_1(v)^2 = Q_1(v) \cdot 1$ *koşulunu sağlayan*

$$j_1 : (V, Q_1) \to Cl(V, Q_1)$$

lineer dönüşümü vardır. Aynı şekilde (V, Q_2) *kuadratik uzayına karşılık gelen Clifford cebiri* $Cl(V, Q_2)$ *olduğundan her* $v \in V$ *için* $j_2(v)^2 = Q_2(v) \cdot 1$ *koşulunu sağlayan*

$$j_2 : (V, Q_2) \to Cl(V, Q_2)$$

lineer dönüşümü vardır. Q_1 *ve* Q_2 *kuadratik formları denk olduğundan* $\sigma : (V, Q_1) \to (V, Q_2)$ *birebir örten lineer dönüşümü vardır öyle ki her* $v \in V$ *için* $Q_1(v) = Q_2(\sigma(v))$ *koşulu sağlanır.*

$$u = (j_2 \circ \sigma) : (V, Q_1) \to Cl(V, Q_2)$$

lineer dönüşümünü alalım. Her $v \in V$ *için*

$$\begin{aligned} u(v)^2 &= (j_2 \circ \sigma)(v)(j_2 \circ \sigma)(v) = j_2(\sigma(v))^2 \\ &= Q_2(\sigma(v)) \cdot 1 = Q_1(v) \cdot 1 \end{aligned}$$

dir. Bu durumda Clifford cebirinin evrensel özelliğinden diagramı değişmeli yapacak şekilde (yani $U \circ j_1 = u$*) tek bir* $U : Cl(V, Q_1) \to Cl(V, Q_2)$ *cebir homomorfizmi vardır.* $\sigma : (V, Q_1) \to (V, Q_2)$ *dönüşümü, birebir örten lineer dönüşüm olduğundan* $\sigma^{-1} : (V, Q_2) \to (V, Q_1)$ *yazabiliriz ve her* $v \in V$ *için* $Q_2(v) = Q_1(\sigma^{-1}(v))$ *dir.*

$$u' = \left(j_1 \circ \sigma^{-1}\right) : (V, Q_2) \to Cl(V, Q_1)$$

dönüşümü lineer dönüşümdür ve her $v \in V$ *için*

$$\begin{aligned} u'(v)^2 &= \left(j_1 \circ \sigma^{-1}\right)(v)\left(j_1 \circ \sigma^{-1}\right)(v) = j_1\left(\sigma^{-1}(v)\right)^2 \\ &= Q_1\left(\sigma^{-1}(v)\right) \cdot 1 = Q_2(v) \cdot 1 \end{aligned}$$

eşitliği sağlanır.

Bu durumda Clifford cebirinin evrensel özelliğinden diagramı değişmeli yapacak şekilde (yani $U' \circ j_2 = u'$) tek bir $U' : Cl(V, Q_2) \rightarrow Cl(V, Q_1)$ cebir homomorfizmi vardır.

$$\begin{array}{ccc} (V,Q_2) & \stackrel{j_2}{\rightarrow} & Cl(V,Q_2) \\ & j_2 \searrow & \downarrow I \\ & & Cl(V,Q_2) \end{array} \qquad \begin{array}{ccc} (V,Q_2) & \stackrel{j_2}{\rightarrow} & Cl(V,Q_2) \\ & j_2 \searrow & \downarrow U \circ U' \\ & & Cl(V,Q_2) \end{array}$$

Yukarıdaki diagramları değişmeli yapacak şekildeki $Cl(V, Q_2) \rightarrow Cl(V, Q_2)$ dönüşümü tek olmak zorunda olduğundan $U \circ U' = U' \circ U = I$ olmalıdır. Bu durumda U ve U' dönüşümleri, biri diğerinin tersi olmaktadır. Dolayısıyla U bir izomorfizmdir. ■

4 $\mathbb{R}^n$ ÜZERİNDEKİ DEJENERE OLMAYAN CLİFFORD CEBİRLERİ

Bu bölümde $V = \mathbb{R}^n$ reel vektör uzayını ve bu uzay üzerinde tanımlanan $Q : \mathbb{R}^n \to \mathbb{R}$ dejenere olmayan kuadratik formları alacağız. Bunun için de

$$Q(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \ (n = p + q)$$

kuadratik formunu gözönüne almak yeterlidir. Bundan böyle $\mathbb{R}^n$ üzerinde $Q(x)$ kuadratik formuna karşılık gelen Clifford cebrini de $Cl_{p,q}$ şeklinde göstereceğiz.

Şimdi Önerme (3.5) yi kullanarak sırasıyla $Cl_{0,1}, Cl_{0,2}, Cl_{1,0}, Cl_{2,0}$ ve $Cl_{1,1}$ Clifford cebirlerini hesaplayalım. Yani bilinen hangi cebirlere izomorf olduğunu belirleyelim.

1) $\mathbb{R}$ üzerindeki dejenere olmayan formlara karşılık gelen Clifford cebirleri:

i) $Q(x) = x^2$ kuadratik formu alındığında $(\mathbb{R}, Q)$ kuadratik uzayına karşılık gelen Clifford cebrini $Cl_{1,0}$ ile gösteriyoruz. $\{e\}$ bu kuadratik uzay için Sylvester tabanı olsun. O zaman $Q(e) = 1$ olup $e^2 = Q(e) \cdot 1 = 1$ dir ve Önerme (3.5) den dolayı $Cl_{1,0} = span\{1, e\}$ olur. Herhangi bir $\alpha \in Cl_{1,0}$ elemanı $\alpha_0, \alpha_1 \in \mathbb{R}$ için $\alpha = \alpha_0 1 + \alpha_1 e$ şeklinde yazılabilir. Bu durumda, tabanlar üzerinde

$$\begin{array}{rccc} \Psi_{1,0}: & Cl_{1,0} & \to & \mathbb{R} \oplus \mathbb{R} \\ & 1 & \mapsto & (1,1) \\ & e & \mapsto & (-1,1) \end{array}$$

şeklinde tanımlanan $\Psi_{1,0}$ dönüşümü bir cebir izomorfizmidir. Bu izomorfizmi daha açık olarak

$$\Psi_{1,0}(\alpha_0 1 + \alpha_1 e) = (\alpha_0 - \alpha_1, \alpha_0 + \alpha_1)$$

şeklinde yazabiliriz. O halde $Cl_{1,0}$ cebri $\mathbb{R} \oplus \mathbb{R}$ cebrine izomorftur.

ii) $Q(x) = -x^2$ kuadratik formu alındığında $(\mathbb{R}, Q)$ kuadratik uzayına karşılık gelen Clifford cebrini $Cl_{0,1}$ ile gösteriyoruz. $\{\varepsilon\}$ bu kuadratik uzay için Sylvester tabanı olsun. O zaman $Q(\varepsilon) = -1$ olup $\varepsilon^2 = Q(\varepsilon) \cdot 1 = -1$ dir ve Önerme (3.5) den dolayı $Cl_{0,1} = span\{1, \varepsilon\}$ olur. Herhangi bir $\alpha \in Cl_{0,1}$ elemanı $\alpha_0, \alpha_1 \in \mathbb{R}$ için $\alpha = \alpha_0 1 + \alpha_1 \varepsilon$ şeklinde yazılabilir.

Bu durumda, tabanlar üzerinde

$$\begin{array}{rcl} \Psi_{0,1}: Cl_{0,1} & \rightarrow & \mathbb{C} \\ 1 & \mapsto & 1 \\ \varepsilon & \mapsto & i \end{array}$$

şeklinde tanımlanan $\Psi_{0,1}$ dönüşümü bir cebir izomorfizmidir. $\Psi_{0,1}$ dönüşümünün açık ifadesini de

$$\Psi_{0,1}(\alpha_0 1 + \alpha_1 \varepsilon) = \alpha_0 + \alpha_1 i$$

şeklinde yazabiliriz. O halde $Cl_{0,1}$ cebri $\mathbb{C}$ kompleks sayılar cebrine izomorftur.

2) $\mathbb{R}^2$ üzerindeki dejenere olmayan formlara karşılık gelen Clifford cebirleri:

i) $Q(x) = x_1^2 + x_2^2$ formu gözönüne alındığında karşılık gelen Clifford cebri $Cl_{2,0}$ dır. $\{e_1, e_2\}$ bu kuadratik uzay için Sylvester tabanı olsun. O zaman $Q(e_i) = 1$ olup $e_i^2 = Q(e_i) \cdot 1 = 1$ dir ve Önerme (3.5) den dolayı

$$Cl_{2,0} = span\,\{1, e_1, e_2, e_1e_2\}$$

olur. Herhangi bir $\alpha \in Cl_{2,0}$ elemanı $\alpha_0, \alpha_1, \alpha_2, \alpha_{12} \in \mathbb{R}$ için

$$\alpha = \alpha_0 1 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_{12} e_1 e_2$$

şeklinde yazılabilir. Bu durumda, tabanlar üzerinde

$$\begin{array}{rcl} \Psi_{2,0}: Cl_{2,0} & \rightarrow & \mathbb{R}(2) \\ 1 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ e_1 & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ e_2 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ e_1e_2 & \mapsto & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{array}$$

şeklinde tanımlanan $\Psi_{2,0}$ dönüşümü bir cebir izomorfizmidir. Bu dönüşümün daha açık ifadesi de aşağıdaki şeklindedir.

$$\Psi_{2,0}(\alpha_0 1 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_{12} e_1 e_2) = \begin{bmatrix} \alpha_0 + \alpha_2 & \alpha_1 - \alpha_{12} \\ \alpha_1 + \alpha_{12} & \alpha_0 - \alpha_2 \end{bmatrix}$$

Sonuç olarak $Cl_{2,0}$ cebri $\mathbb{R}(2)$ cebrine izomorftur.

ii) $Q(x) = x_1^2 - x_2^2$ formu gözönüne alındığında karşılık gelen Clifford cebri $Cl_{1,1}$ dır. $\{e, \varepsilon\}$ bu kuadratik uzay için Sylvester tabanı olsun. O zaman $Q(e) = 1$ olup $e^2 = Q(e) \cdot 1 = 1$ ve $Q(\varepsilon) = -1$ olup $\varepsilon^2 = Q(\varepsilon) \cdot 1 = -1$ dir. Önerme (3.5) den dolayı $Cl_{1,1} = span\{1, e, \varepsilon, e\varepsilon\}$ olur. Herhangi bir $\alpha \in Cl_{1,1}$ elemanı $\alpha_0, \alpha_1, \alpha_2, \alpha_{12} \in \mathbb{R}$ için $\alpha = \alpha_0 1 + \alpha_1 e + \alpha_2 \varepsilon + \alpha_{12} e\varepsilon$ şeklinde yazılabilir. Bu durumda, tabanlar üzerinde

$$\begin{aligned}
\Psi_{1,1} : \quad Cl_{1,1} \quad &\longrightarrow \quad \mathbb{R}(2) \\
1 \quad &\longmapsto \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
e \quad &\longmapsto \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\varepsilon \quad &\longmapsto \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
e\varepsilon \quad &\longmapsto \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}$$

şeklinde tanımlanan $\Psi_{1,1}$ dönüşümü bir cebir izomorfizmidir. Bu dönüşümün daha açık ifadesi de

$$\Psi_{1,1}(\alpha_0 1 + \alpha_1 e + \alpha_2 \varepsilon + \alpha_{12} e\varepsilon) = \begin{bmatrix} \alpha_0 + \alpha_{12} & \alpha_1 - \alpha_2 \\ \alpha_1 + \alpha_2 & \alpha_0 - \alpha_{12} \end{bmatrix}$$

şeklindedir. Sonuç olarak $Cl_{1,1}$ cebri $\mathbb{R}(2)$ cebrine izomorftur.

iii) $Q(x) = -x_1^2 - x_2^2$ formu gözönüne alındığında karşılık gelen Clifford cebri $Cl_{0,2}$ dır. $\{\varepsilon_1, \varepsilon_2\}$ bu kuadratik uzay için Sylvester tabanı olsun. O zaman $Q(\varepsilon_i) = -1$ olup $\varepsilon_i^2 = Q(\varepsilon_i) \cdot 1 = -1$ dir. Önerme (3.5) den dolayı

$$Cl_{0,2} = span\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$$

olur. Herhangi bir $\alpha \in Cl_{0,2}$ elemanı $\alpha_0, \alpha_1, \alpha_2, \alpha_{12} \in \mathbb{R}$ için

$$\alpha = \alpha_0 1 + \alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2 + \alpha_{12} \varepsilon_1 \varepsilon_2$$

şeklinde yazılabilir.

Bu durumda, tabanlar üzerinde

$$\begin{array}{rcl} \Psi_{0,2}: \ Cl_{0,2} & \longrightarrow & \mathbb{H} \\ 1 & \mapsto & 1 \\ \varepsilon_1 & \mapsto & i \\ \varepsilon_2 & \mapsto & j \\ \varepsilon_1\varepsilon_2 & \mapsto & k \end{array}$$

şeklinde tanımlanan $\Psi_{0,2}$ dönüşümü bir cebir izomorfizmidir. Bu dönüşümün daha açık ifadesi de

$$\Psi_{0,2}(\alpha_0 1 + \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_{12}\varepsilon_1\varepsilon_2) = \alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_{12} k$$

dir. Sonuç olarak $Cl_{0,2}$ cebri $\mathbb{H}$ cebrine izomorftur.

Yukarıdaki örneklerde görüldüğü gibi boyut büyüdükçe, bu yolla ilgili Clifford cebrinin bilinen hangi cebire izomorf olduğunu belirlemek güçleşmektedir. Bu nedenle farklı boyutlardaki Clifford cebirleri arasında indirgeme ilişkisi veren aşağıdaki üç temel izomorfizm bizim için faydalı olacak.

Önerme 4.1 1)$Cl_{0,n+2} \cong Cl_{n,0} \otimes Cl_{0,2}$

2)$Cl_{n+2,0} \cong Cl_{0,n} \otimes Cl_{2,0}$

3)$Cl_{p+1,q+1} \cong Cl_{p,q} \otimes Cl_{1,1}$

Kanıt. Bu izomorfizmler sırasıyla aşağıdaki gibi tanımlanır.

$\pi_1 : Cl_{0,n+2} \longrightarrow Cl_{n,0} \otimes Cl_{0,2}$

$$\pi_1(\varepsilon_i) = \begin{cases} e_{i-2} \otimes \varepsilon_1\varepsilon_2 & 3 \le i \le n+2 \text{ ise}, \\ 1 \otimes \varepsilon_i & i = 1 \text{ ve } i = 2 \text{ ise}, \end{cases}$$

$\pi_2 : Cl_{n+2,0} \longrightarrow Cl_{0,n} \otimes Cl_{2,0}$

$$\pi_2(e_i) = \begin{cases} \varepsilon_{i-2} \otimes e_1 e_2 & 3 \le i \le n+2 \text{ ise}, \\ 1 \otimes e_i & i = 1 \text{ ve } i = 2 \text{ ise}, \end{cases} \quad \text{ve}$$

$\pi_3 : Cl_{p+1,q+1} \longrightarrow Cl_{p,q} \otimes Cl_{1,1}$

$$\pi_3(e_i) = \begin{cases} e_i \otimes e_1\varepsilon_1 & 1 \le i \le p \text{ ise}, \\ 1 \otimes e_1 & i = p+1 \text{ ise}, \end{cases}$$

$$\pi_3(\varepsilon_j) = \begin{cases} \varepsilon_j \otimes e_1\varepsilon_1 & 1 \le j \le q \text{ ise}, \\ 1 \otimes \varepsilon_1 & j = q+1 \text{ ise}, \end{cases} \quad \blacksquare$$

Yukarıdaki π_1, π_2 izomorfizmleri yardımıyla da $Cl_{0,n+4}$ cebri ile $Cl_{0,n} \otimes Cl_{2,0} \otimes Cl_{0,2}$ cebri arasında aşağıdaki izomorfizm elde edilir ve bu izomorfizm dörtlü bir indirgeme formülü verir.

$$\begin{array}{llllll}
\pi : & Cl_{0,n+4} & \longrightarrow & Cl_{n+2,0} \otimes Cl_{0,2} & \longrightarrow & Cl_{0,n} \otimes Cl_{2,0} \otimes Cl_{0,2} \\
& \varepsilon_1 & \longmapsto & 1 \otimes \varepsilon_1 & \longmapsto & 1 \otimes 1 \otimes \varepsilon_1 \\
& \varepsilon_2 & \longmapsto & 1 \otimes \varepsilon_2 & \longmapsto & 1 \otimes 1 \otimes \varepsilon_2 \\
& \varepsilon_3 & \longmapsto & e_1 \otimes \varepsilon_1\varepsilon_2 & \longmapsto & 1 \otimes e_1 \otimes \varepsilon_1\varepsilon_2 \\
& \varepsilon_4 & \longmapsto & e_2 \otimes \varepsilon_1\varepsilon_2 & \longmapsto & 1 \otimes e_2 \otimes \varepsilon_1\varepsilon_2 \\
& \varepsilon_5 & \longmapsto & e_3 \otimes \varepsilon_1\varepsilon_2 & \longmapsto & \varepsilon_1 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \\
& \varepsilon_6 & \longmapsto & e_4 \otimes \varepsilon_1\varepsilon_2 & \longmapsto & \varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 \\
& \vdots & \longmapsto & \vdots & \longmapsto & \vdots \\
& \varepsilon_{n+4} & \longmapsto & e_{n+2} \otimes \varepsilon_1\varepsilon_2 & \longmapsto & \varepsilon_n \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2
\end{array}$$

Bu son izomorfizm $Cl_{0,8}$ e iki kere uygulanırsa

$$Cl_{0,8} \cong Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2}$$

elde edilir. Keyfi bir n için $Cl_{0,n+8}$ e iki defa π uygulanırsa ve

$$Cl_{0,8} \cong Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2}$$

olduğu kullanılırsa, $Cl_{0,m}$ tipindeki Clifford cebirleri arasında

$$Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8}$$

şeklinde 8-li bir indirgeme formülü elde edilir.

Benzer işlemler yapılarak $Cl_{m,0}$ tipindeki Clifford cebirleri arasında da $Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0}$ şeklinde 8-li bir indirgeme formülü elde edilir.

Yukarıdaki izomorfizmler kullanılarak tüm Clifford cebirleri hesaplanabilir. Ancak bu hesaplamalar sırasında reel cebirler arasındaki aşağıdaki izomorfizmler gerekmektedir. Aşağıda geçen tensör çarpımlarının hepsi $\mathbb{R}$ reel sayılar cismi üzerinden yapılmaktadır.

1. $\mathbb{K} = \mathbb{R}, \mathbb{C}$ veya $\mathbb{H}$ olmak üzere $\mathbb{R}(n) \otimes \mathbb{K} \cong \mathbb{K}(n)$ dir ve bu izomorfizm $[a_{ij}] \otimes k \longmapsto [ka_{ij}]$ şeklinde verilir.

2. $\mathbb{R}(n)\otimes\mathbb{R}(m)\cong\mathbb{R}(nm)$ dir ve bu izomorfizm $A\otimes B\longmapsto(a_{ij}B)$ şeklinde verilir. (Bu eşlemeye iki matrisin tensör çarpımı bazen de Kronecker çarpımı denir.)

3. $\mathbb{C}\otimes\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$ dir ve bu izomorfizm $z\otimes w\longmapsto(zw,\bar{z}w)$ şeklinde verilir.

4. $\mathbb{C}\otimes\mathbb{H}\cong\mathbb{C}(2)$ dir ve bu izomorfizm

$$\begin{array}{rcl} f: \mathbb{C}\otimes\mathbb{H} & \rightarrow & End_{\mathbb{C}}(\mathbb{H})\cong\mathbb{C}(2) \\ z\otimes q & \mapsto & f_{z,q} \end{array}$$

şeklindedir ve bu dönüşüm her $x\in\mathbb{H}$ için $f_{z,q}(x)=zx\bar{q}$ şeklinde tanımlanır. İleride $\mathbb{C}\otimes\mathbb{H}$ nın taban vektörlerinin f altındaki görüntülerine ihtiyacımız olacak. Bu nedenle $z\in\{1,i\}$ ve $q\in\{1,i,j,k\}$ için $f_{z,q}$ ların $\mathbb{H}$ nın $\{1,j\}$ kompleks tabanına göre matrislerini bulalım.

$$\left\{\begin{array}{l} f_{1,1}(1)=1.1.1=(1)\,1 \\ f_{1,1}(j)=1.j.1=(1)\,j \end{array}\right\} \Longrightarrow f_{1,1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left\{\begin{array}{l} f_{1,i}(1)=1.1.(-i)=(-i)\,1 \\ f_{1,i}(j)=1.j.(-i)=(i)\,j \end{array}\right\} \Longrightarrow f_{1,i}=\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$\left\{\begin{array}{l} f_{1,j}(1)=1.1.(-j)=(-1)\,j \\ f_{1,j}(j)=1.j.(-j)=(1)\,1 \end{array}\right\} \Longrightarrow f_{1,j}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\left\{\begin{array}{l} f_{1,k}(1)=1.1.(-k)=(-i)\,j \\ f_{1,k}(j)=1.j.(-k)=(-i)\,1 \end{array}\right\} \Longrightarrow f_{1,k}=\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

$$\left\{\begin{array}{l} f_{i,1}(1)=i.1.1=(i)\,1 \\ f_{i,1}(j)=i.j.1=(i)\,j \end{array}\right\} \Longrightarrow f_{i,1}=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

$$\left\{\begin{array}{l} f_{i,i}(1)=i.1.(-i)=(1)\,1 \\ f_{i,i}(j)=i.j.(-i)=(-1)\,j \end{array}\right\} \Longrightarrow f_{i,i}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\left\{\begin{array}{l} f_{i,k}(1)=i.1.(-k)=(1)\,j \\ f_{i,k}(j)=i.j.(-k)=(1)\,1 \end{array}\right\} \Longrightarrow f_{i,k}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

5. $\mathbb{H}\otimes\mathbb{H}\cong\mathbb{R}(4)$ dir ve bu izomorfizm

$$\begin{array}{rcl} \Phi: \mathbb{H}\otimes\mathbb{H} & \rightarrow & End_{\mathbb{R}}(\mathbb{H})\cong\mathbb{R}(4) \\ q_1\otimes q_2 & \mapsto & \Phi_{q_1,q_2} \end{array}$$

şeklindedir ve Φ_{q_1,q_2} dönüşümü de her $q \in \mathbb{H}$ için $\Phi_{q_1,q_2}(q) = q_1 q \bar{q}_2$ şeklinde tanımlanır. Şimdi her $q_1, q_2 \in \{1, i, j, k\}$ için Φ_{q_1,q_2} lerin $\mathbb{H} \cong \mathbb{R}^4$ uzayının $\{e_1, e_2, e_3, e_4\}$ standart tabanına göre matrislerini bulalım. Yapılan işlemler sonucunda bu işlemler aşağıdaki şekildedir.

$$\Phi_{1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Phi_{i,1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Phi_{1,i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Phi_{i,i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Phi_{1,j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \Phi_{i,j} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_{1,k} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad \Phi_{i,k} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\Phi_{j,1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad \Phi_{k,1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_{j,i} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \Phi_{k,i} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\Phi_{j,j} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \Phi_{k,j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Phi_{j,k} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \Phi_{k,k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Daha önceden $\mathbb{R}$ ve $\mathbb{R}^2$ üzerindeki tüm dejenere olmayan formlara karşılık gelen Clifford cebirlerini bulmuştuk. Şimdi yukarıda verilen bilgiler kullanılarak diğer Clifford cebirleri de hesaplanabilir.

Önerme (4.1) den dolayı $Cl_{0,3} \cong Cl_{1,0} \otimes Cl_{0,2}$ dir. Daha önceden $Cl_{1,0} \cong \mathbb{R} \oplus \mathbb{R}$, $Cl_{2,0} \cong \mathbb{R}(2)$ ve $Cl_{0,2} \cong \mathbb{H}$ izomorfizmleri elde edilmişti. Bunlar kullanılırsa,

$$\begin{aligned}
Cl_{0,3} &\cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \cong (\mathbb{R} \otimes \mathbb{H}) \oplus (\mathbb{R} \otimes \mathbb{H}) \cong \mathbb{H} \oplus \mathbb{H} \\
Cl_{0,4} &\cong Cl_{2,0} \otimes Cl_{0,2} \cong \mathbb{R}(2) \otimes \mathbb{H} \cong \mathbb{H}(2) \\
Cl_{0,5} &\cong Cl_{0,1} \otimes Cl_{2,0} \otimes Cl_{0,2} \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{H} \cong \mathbb{C} \otimes (\mathbb{H} \otimes \mathbb{R}(2)) \\
&\cong (\mathbb{C} \otimes \mathbb{H}) \otimes \mathbb{R}(2) \cong \mathbb{C}(2) \otimes \mathbb{R}(2) \cong (\mathbb{C} \otimes \mathbb{R}(2)) \otimes \mathbb{R}(2) \\
&\cong \mathbb{C} \otimes (\mathbb{R}(2) \otimes \mathbb{R}(2)) \cong \mathbb{C} \otimes \mathbb{R}(4) \cong \mathbb{C}(4)
\end{aligned}$$

$Cl_{0,3} \cong \mathbb{H} \oplus \mathbb{H}$, $Cl_{0,4} \cong \mathbb{H}(2)$ ve $Cl_{0,5} \cong \mathbb{C}(4)$ izomorfizmaları elde edilir. Benzer şekilde,

$$\begin{aligned}
Cl_{0,6} &\cong Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \cong \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \\
&\cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
Cl_{0,7} &\cong Cl_{0,3} \otimes Cl_{2,0} \otimes Cl_{0,2} \cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{R}(2) \otimes \mathbb{H} \\
&\cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong (\mathbb{H} \otimes \mathbb{H} \oplus \mathbb{H} \otimes \mathbb{H}) \otimes \mathbb{R}(2) \\
&\cong (\mathbb{R}(4) \oplus \mathbb{R}(4)) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \oplus \mathbb{R}(8) \\
Cl_{0,8} &\cong Cl_{0,4} \otimes Cl_{2,0} \otimes Cl_{0,2} \cong Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \\
&\cong \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \cong \mathbb{R}(2) \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{H} \\
&\cong \mathbb{R}(4) \otimes \mathbb{R}(4) \cong \mathbb{R}(16)
\end{aligned}$$

$Cl_{0,6} \cong \mathbb{R}(8)$, $Cl_{0,7} \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$ ve $Cl_{0,8} \cong \mathbb{R}(16)$ izomorfizmaları elde edilir.

Şimdi de $3 \leq n \leq 8$ için $Cl_{n,0}$ Clifford cebirlerinin izomorfizmlerini Önerme (4.1) yi kullanarak hesaplayalım.

Daha önceden $Cl_{0,1} \cong \mathbb{C}$, $Cl_{0,2} \cong \mathbb{H}$ ve $Cl_{2,0} \cong \mathbb{R}(2)$ izomorfizmleri elde edilmişti. Bunlar kullanılırsa,

$$
\begin{aligned}
Cl_{3,0} &\cong Cl_{0,1} \otimes Cl_{2,0} \cong \mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2) \\
Cl_{4,0} &\cong Cl_{0,2} \otimes Cl_{2,0} \cong \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2) \\
Cl_{5,0} &\cong Cl_{1,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{R}(2) \\
&\cong (\mathbb{R} \oplus \mathbb{R}) \otimes (\mathbb{R}(2) \otimes \mathbb{H}) \cong (\mathbb{R}(2) \oplus \mathbb{R}(2)) \otimes \mathbb{H} \\
&\cong (\mathbb{R}(2) \otimes \mathbb{H}) \otimes (\mathbb{R}(2) \otimes \mathbb{H}) \cong (\mathbb{H} \otimes \mathbb{R}(2)) \otimes (\mathbb{H} \otimes \mathbb{R}(2)) \\
&\cong \mathbb{H}(2) \oplus \mathbb{H}(2)
\end{aligned}
$$

$Cl_{3,0} \cong \mathbb{C}(2)$, $Cl_{4,0} \cong \mathbb{H}(2)$ ve $Cl_{5,0} \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$ izomorfizmaları bulunur. Benzer şekilde

$$
\begin{aligned}
Cl_{6,0} &\cong Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \cong \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \\
&\cong \mathbb{H} \otimes \mathbb{R}(4) \cong \mathbb{H}(4) \\
Cl_{7,0} &\cong Cl_{3,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \cong \mathbb{C}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \\
&\cong \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{C}(2) \otimes \mathbb{R}(4) \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{R}(4) \\
&\cong \mathbb{C} \otimes \mathbb{R}(8) \cong \mathbb{C}(8) \\
Cl_{8,0} &\cong Cl_{4,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \cong \mathbb{H}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \\
&\cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \otimes \mathbb{R}(4) \cong \mathbb{R}(16)
\end{aligned}
$$

$Cl_{6,0} \cong \mathbb{H}(4)$, $Cl_{7,0} \cong \mathbb{C}(8)$ ve $Cl_{8,0} \cong \mathbb{R}(16)$ izomorfizmaları elde edilir.

Bu elde edilenleri ve Önerme (4.1) yi kullanarak tüm $Cl_{p,q}$ ları da bulabiliriz. İlk başta $Cl_{1,1} \cong \mathbb{R}(2)$, $Cl_{0,1} \cong \mathbb{C}$ ve $Cl_{1,0} \cong \mathbb{R} \oplus \mathbb{R}$ izomorfizmlerini elde etmiştik. Bunlar kullanılırsa aşağıdakiler elde edilir.

$$
\begin{aligned}
Cl_{2,1} &\cong Cl_{1,0} \otimes Cl_{1,1} \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{R}(2) \cong \mathbb{R}(2) \oplus \mathbb{R}(2) \\
Cl_{1,2} &\cong Cl_{0,1} \otimes Cl_{1,1} \cong \mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2) \\
Cl_{3,1} &\cong Cl_{2,0} \otimes Cl_{1,1} \cong \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \\
Cl_{2,2} &\cong Cl_{1,1} \otimes Cl_{1,1} \cong \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \\
Cl_{1,3} &\cong Cl_{0,2} \otimes Cl_{1,1} \cong \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2) \\
Cl_{1,4} &\cong Cl_{0,3} \otimes Cl_{1,1} \cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{R}(2) \cong \mathbb{H}(2) \oplus \mathbb{H}(2) \\
Cl_{2,3} &\cong Cl_{1,2} \otimes Cl_{1,1} \cong \mathbb{C}(2) \otimes \mathbb{R}(2) \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \\
&\cong \mathbb{C} \otimes \mathbb{R}(4) \cong \mathbb{C}(4) \\
Cl_{3,2} &\cong Cl_{2,1} \otimes Cl_{1,1} \cong (\mathbb{R}(2) \oplus \mathbb{R}(2)) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \oplus \mathbb{R}(4) \\
Cl_{4,1} &\cong Cl_{3,0} \otimes Cl_{1,1} \cong \mathbb{C}(2) \otimes \mathbb{R}(2) \cong \mathbb{C} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \\
&\cong \mathbb{C} \otimes \mathbb{R}(4) \cong \mathbb{C}(4)
\end{aligned}
$$

Bu şekilde devam ettirirsek aşağıdaki tabloyu elde ederiz.

q / p	0	1	2	3	4	5	6	7	8
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(2^5)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(2^5)$	$\mathbb{C}(2^5)$	$\mathbb{H}(2^5)$	$\mathbb{H}(2^5)\oplus\mathbb{H}(2^5)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(2^5)$	$\mathbb{R}(2^5)\oplus\mathbb{R}(2^5)$	$\mathbb{R}(2^6)$	$\mathbb{C}(2^6)$	$\mathbb{H}(2^6)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(2^5)$	$\mathbb{R}(2^6)$	$\mathbb{R}(2^6)\oplus\mathbb{R}(2^6)$	$\mathbb{R}(2^7)$	$\mathbb{C}(2^7)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(2^5)$	$\mathbb{C}(2^6)$	$\mathbb{R}(2^7)$	$\mathbb{R}(2^7)\oplus\mathbb{R}(2^7)$	$\mathbb{R}(2^8)$

$Cl_{p,q}$ Clifford cebrinin izomorfizm tablosu

Dikkat edilirse yukarıdaki tablo oluşturulurken birkaç tanesi dışında izomorfizmlerin açık ifadeleri gerekmedi. Bu tablodan ve periyodiklik ilişkilerinden yararlanarak herhangi $Cl_{p,q}$ Clifford cebrinin bilinen hangi matris cebrine izomorf olduğunu, arada açık bir izomorfizm vermeksizin, tespit etmek mümkündür. Örneğin $\mathbb{R}^{21}$ üzerindeki $Cl_{7,14}$ Clifford cebrinin bilinen hangi matris cebrine izomorf olduğunu kolayca bulabiliriz. Periyodiklik ilişkisinden

$$Cl_{7,14} = Cl_{7,6+8} \cong Cl_{7,6} \otimes Cl_{0,8}$$

dir, bu durumda tabloya bakıldığında $Cl_{7,6} \cong \mathbb{R}(2^6) \oplus \mathbb{R}(2^6)$ (7. satır ve 6. sütundan) ve $Cl_{0,8} \cong \mathbb{R}(16)$ (0. satır ve 8. sütundan) olduğu görülür. Bunlar yerine yazılırsa

$$Cl_{7,14} \cong \left(\mathbb{R}\left(2^6\right) \oplus \mathbb{R}\left(2^6\right)\right) \otimes \mathbb{R}(16) \cong \mathbb{R}\left(2^{10}\right) \oplus \mathbb{R}\left(2^{10}\right)$$

elde edilir. Benzer hesaplamalarla diğer $Cl_{p,q}$ Clifford cebirleri için aşağıdaki tablo elde edilir.

$p-q \pmod 8$	$Cl_{p,q}$
0	$\mathbb{R}\left(2^l\right)$
1	$\mathbb{R}\left(2^l\right) \oplus \mathbb{R}\left(2^l\right)$
2	$\mathbb{R}\left(2^l\right)$
3	$\mathbb{C}\left(2^l\right)$
4	$\mathbb{H}\left(2^{l-1}\right)$
5	$\mathbb{H}\left(2^{l-1}\right) \oplus \mathbb{H}\left(2^{l-1}\right)$
6	$\mathbb{H}\left(2^{l-1}\right)$
7	$\mathbb{C}\left(2^l\right)$

Burada l sayısı $\frac{p+q}{2}$ nin tam kısmıdır.

Bizim amacımız herhangi $Cl_{p,q}$ Clifford cebri verildiğinde ilgili matris cebrine giden izomorfizmin açık olarak elde edilmesi için bir yöntem vermektir. Bunun için önce $Cl_{0,n}$ tipindeki Clifford cebrinden ilgili matris cebrine giden izomorfizmi bulacağız (bu tip izomorfizmler [5] de verilmiştir).

Sonra $\pi_2 : Cl_{n+2,0} \to Cl_{0,n} \otimes Cl_{2,0}$ izomorfizminden yararlanarak $Cl_{n,0}$ tipindeki tüm Clifford cebirleri için ilgili izomorfizmleri elde edeceğiz. $Cl_{0,n}$ ve $Cl_{n,0}$ ler için izomorfizmler elde edildikten sonra $\pi_3 : Cl_{p+1,q+1} \to Cl_{p,q} \otimes Cl_{1,1}$ izomorfizmini kullanarak da tüm $Cl_{p,q}$ Clifford cebirleri için de söz konusu izomorfizmleri elde edeceğiz. $Cl_{0,n}$ Clifford cebirleri için izomorfizmleri önce $1 \leq n \leq 8$ için, sonrada bunlar yardımıyla $n > 8$ için sözkonusu izomorfizmleri elde edeceğiz.

4.1 $Cl_{0,n}$ Clifford cebirleri için izomorfizmlerin elde edilmesi

1) $1 \leq n \leq 8$ için izomorfizmlerin elde edilmesi (Tabloda 0. satır)

i) $n = 1$ için $\Psi_{0,1} : Cl_{0,1} \to \mathbb{C}$ şeklinde bulmuştuk.

ii) $n = 2$ için $\Psi_{0,2} : Cl_{0,2} \to \mathbb{H}$ şeklinde bulmuştuk.

iii) $n = 3$ için $\Psi_{0,3} : Cl_{0,3} \to \mathbb{H} \oplus \mathbb{H}$ izomorfizmini,daha önce bulduğumuz $\Psi_{1,0} : Cl_{1,0} \to \mathbb{R} \oplus \mathbb{R}$ ve $\Psi_{0,2} : Cl_{0,2} \to \mathbb{H}$ izomorfizmlerinden yararlanarak bulalım.

$$\begin{array}{ccccccc}
Cl_{0,3} & \to & Cl_{1,0} \otimes Cl_{0,2} & \to & (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} & \to & \mathbb{H} \oplus \mathbb{H} \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & (1,1) \otimes i & \mapsto & (i,i) \\
\varepsilon_2 & \mapsto & 1 \otimes \varepsilon_2 & \mapsto & (1,1) \otimes j & \mapsto & (j,j) \\
\varepsilon_3 & \mapsto & \varepsilon_1 \otimes \varepsilon_1\varepsilon_2 & \mapsto & (1,-1) \otimes k & \mapsto & (k,-k)
\end{array}$$

iv) $n = 4$ için $\Psi_{0,4} : Cl_{0,4} \to \mathbb{H}(2)$ izomorfizmini daha önce bulduğumuz $\Psi_{2,0} : Cl_{2,0} \to \mathbb{R}(2)$ ve $\Psi_{0,2} : Cl_{0,2} \to \mathbb{H}$ izomorfizmlerinden yararlanarak bulalım.

$$\begin{array}{llllllll}
Cl_{0,4} & \to & Cl_{2,0}\otimes Cl_{0,2} & \to & \mathbb{R}(2)\otimes\mathbb{H} & & \to & \mathbb{H}(2) \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & \begin{bmatrix}1&0\\0&1\end{bmatrix} & \otimes i & \mapsto & \begin{bmatrix}i&0\\0&i\end{bmatrix} \\
\varepsilon_2 & \mapsto & 1\otimes\varepsilon_2 & \mapsto & \begin{bmatrix}1&0\\0&1\end{bmatrix} & \otimes j & \mapsto & \begin{bmatrix}j&0\\0&j\end{bmatrix} \\
\varepsilon_3 & \mapsto & e_1\otimes\varepsilon_1\varepsilon_2 & \mapsto & \begin{bmatrix}0&1\\1&0\end{bmatrix} & \otimes k & \mapsto & \begin{bmatrix}0&k\\k&0\end{bmatrix} \\
\varepsilon_4 & \mapsto & e_2\otimes\varepsilon_1\varepsilon_2 & \mapsto & \begin{bmatrix}1&0\\0&-1\end{bmatrix} & \otimes k & \mapsto & \begin{bmatrix}k&0\\0&-k\end{bmatrix}
\end{array}$$

v) $n=5$ için $\Psi_{0,5}:Cl_{0,5}\to\mathbb{C}(4)$ izomorfizminin açık ifadesini, daha önce bulduğumuz izomorfizmlerin açık ifadelerini kullanarak bulalım.

$$\begin{array}{lllll}
Cl_{0,5} & \to & Cl_{0,1}\otimes Cl_{2,0}\otimes Cl_{0,2} & \to & \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{R}(2) \\
\varepsilon_1 & \mapsto & 1\otimes 1\otimes\varepsilon_1 & \mapsto & 1\otimes i\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} \\
\varepsilon_2 & \mapsto & 1\otimes 1\otimes\varepsilon_2 & \mapsto & 1\otimes j\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} \\
\varepsilon_3 & \mapsto & 1\otimes e_1\otimes\varepsilon_1\varepsilon_2 & \mapsto & 1\otimes k\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix} \\
\varepsilon_4 & \mapsto & 1\otimes e_2\otimes\varepsilon_1\varepsilon_2 & \mapsto & 1\otimes k\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix} \\
\varepsilon_5 & \mapsto & \varepsilon_1\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2 & \mapsto & i\otimes k\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}
\end{array}$$

$\mathbb{C}\otimes\mathbb{H}$ dan $\mathbb{C}(2)$ ye giden izomorfizm altında $1\otimes i, 1\otimes j, 1\otimes k, i\otimes k$ vektörlerinin görüntülerini elde etmiştik. Bunlar kullanılırsa aşağıdakiler elde edilir.

$$
\begin{array}{lll}
\rightarrow & \mathbb{C}(2)\otimes\mathbb{R}(2) & \rightarrow \quad \mathbb{C}\otimes\mathbb{R}(2)\otimes\mathbb{R}(2) \\
\mapsto & \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \mapsto \quad i\otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mapsto & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \mapsto \quad 1\otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mapsto & \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto \quad i\otimes \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\mapsto & \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto \quad i\otimes \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto \quad 1\otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{array}
$$

Son gelinen aşamada $\mathbb{R}(2)\otimes\mathbb{R}(2)$ cebrinin $\mathbb{R}(4)$ cebrine izomorf olduğu kullanılırsa aşağıdaki bulunur.

$$
\begin{array}{llll}
Cl_{0,5} & \rightarrow \quad \mathbb{C}\otimes\mathbb{R}(4) & \rightarrow & \mathbb{C}(4) \\
\varepsilon_1 & \mapsto \quad i\otimes \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \mapsto & \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \\
\varepsilon_2 & \mapsto \quad 1\otimes \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\
\varepsilon_3 & \mapsto \quad i\otimes \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

$$\varepsilon_4 \mapsto i\otimes\begin{bmatrix}0&0&-1&0\\0&0&0&1\\-1&0&0&0\\0&1&0&0\end{bmatrix} \mapsto \begin{bmatrix}0&0&-i&0\\0&0&0&i\\-i&0&0&0\\0&i&0&0\end{bmatrix}$$

$$\varepsilon_5 \mapsto 1\otimes\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix} \mapsto \begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}$$

vi) $n=6$ için $\Psi_{0,6} : Cl_{0,6} \to \mathbb{R}(8)$ izomorfizminin açık ifadesini, daha önce bulduğumuz izomorfizmlerin açık ifadelerini kullanarak bulalım.

$$Cl_{0,6} \to Cl_{0,2}\otimes Cl_{2,0}\otimes Cl_{0,2} \to \mathbb{H}\otimes\mathbb{H}\otimes\mathbb{R}(2)$$

$$\varepsilon_1 \mapsto 1\otimes 1\otimes\varepsilon_1 \mapsto 1\otimes i\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\varepsilon_2 \mapsto 1\otimes 1\otimes\varepsilon_2 \mapsto 1\otimes j\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\varepsilon_3 \mapsto 1\otimes e_1\otimes\varepsilon_1\varepsilon_2 \mapsto 1\otimes k\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

$$\varepsilon_4 \mapsto 1\otimes e_2\otimes\varepsilon_1\varepsilon_2 \mapsto 1\otimes k\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$\varepsilon_5 \mapsto \varepsilon_1\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2 \mapsto i\otimes k\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$$\varepsilon_6 \mapsto \varepsilon_2\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2 \mapsto j\otimes k\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$\mathbb{H}\otimes\mathbb{H}$ dan $\mathbb{R}(4)$ ye giden izomorfizm altında $1\otimes i, 1\otimes j, 1\otimes k, i\otimes k, j\otimes k$ vektörlerinin görüntülerini elde etmiştik. Bunlar kullanılırsa aşağıdakiler elde edilir.

$$
\begin{aligned}
Cl_{0,6} &\to \mathbb{R}(4)\otimes\mathbb{R}(2)\\
\varepsilon_1 &\mapsto \begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
\varepsilon_2 &\mapsto \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\\
\varepsilon_3 &\mapsto \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\
\varepsilon_4 &\mapsto \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix}\otimes\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\\
\varepsilon_5 &\mapsto \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\\
\varepsilon_6 &\mapsto \begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\otimes\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Cl_{0,6} &\rightarrow \mathbb{R}(8) \\
\varepsilon_1 &\mapsto \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \\
\varepsilon_2 &\mapsto \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix} \\
\varepsilon_3 &\mapsto \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{aligned}
$$

$$\varepsilon_4 \quad \mapsto \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_5 \quad \mapsto \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_6 \quad \mapsto \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

vii) $n = 7$ için $\Psi_{0,7} : Cl_{0,7} \to \mathbb{R}(8) \oplus \mathbb{R}(8)$ izomorfizminin açık ifadesini, daha önce bulduğumuz izomorfizmlerin açık ifadelerini kullanarak bulalım.

$$
\begin{array}{llllll}
Cl_{0,7} & \rightarrow & Cl_{0,3}\otimes Cl_{2,0}\otimes Cl_{0,2} & \rightarrow & (\mathbb{H}\oplus\mathbb{H})\otimes\mathbb{H}\otimes\mathbb{R}(2) \\
\varepsilon_1 & \mapsto & 1\otimes 1\otimes\varepsilon_1 & \mapsto & (1,1)\otimes i\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} \\
\varepsilon_2 & \mapsto & 1\otimes 1\otimes\varepsilon_2 & \mapsto & (1,1)\otimes j\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} \\
\varepsilon_3 & \mapsto & 1\otimes e_1\otimes\varepsilon_1\varepsilon_2 & \mapsto & (1,1)\otimes k\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix} \\
\varepsilon_4 & \mapsto & 1\otimes e_2\otimes\varepsilon_1\varepsilon_2 & \mapsto & (1,1)\otimes k\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix} \\
\varepsilon_5 & \mapsto & \varepsilon_1\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2 & \mapsto & (i,i)\otimes k\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix} \\
\varepsilon_6 & \mapsto & \varepsilon_2\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2 & \mapsto & (j,j)\otimes k\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix} \\
\varepsilon_7 & \mapsto & \varepsilon_3\otimes e_1e_2\otimes\varepsilon_1\varepsilon_2 & \mapsto & (k,-k)\otimes k\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}
\end{array}
$$

$\mathbb{H}\otimes\mathbb{H}$ dan $\mathbb{R}(4)$ ye giden izomorfizm altında $1\otimes i, 1\otimes j, 1\otimes k, i\otimes k, j\otimes k, k\otimes k$ vektörlerinin görüntülerini elde etmiştik. Bunlar kullanılırsa aşağıdakiler elde edilir.

$$
\begin{array}{lll}
Cl_{0,7} & \rightarrow & (\mathbb{H}\otimes\mathbb{H}\oplus\mathbb{H}\otimes\mathbb{H})\otimes\mathbb{R}(2) \\
\varepsilon_1 & \mapsto & (1\otimes i, 1\otimes i)\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} \\
\varepsilon_2 & \mapsto & (1\otimes j, 1\otimes j)\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix} \\
\varepsilon_3 & \mapsto & (1\otimes k, 1\otimes k)\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix} \\
\varepsilon_4 & \mapsto & (1\otimes k, 1\otimes k)\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix} \\
\varepsilon_5 & \mapsto & (i\otimes k, i\otimes k)\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}
\end{array}
$$

$$\varepsilon_6 \quad \mapsto \quad (j \otimes k, j \otimes k) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\varepsilon_7 \quad \mapsto \quad (k \otimes k, -k \otimes k) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$Cl_{0,7} \rightarrow (\mathbb{R}(4) \oplus \mathbb{R}(4)) \otimes \mathbb{R}(2)$

$$\Psi_{0,7}(\varepsilon_1) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Psi_{0,7}(\varepsilon_2) = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Psi_{0,7}(\varepsilon_3) = \left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Psi_{0,7}(\varepsilon_4) = \left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Psi_{0,7}(\varepsilon_5) = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Psi_{0,7}(\varepsilon_6) = \left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Psi_{0,7}(\varepsilon_7) = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Burada doğal dağılma dönüşümü kullanalım.
$Cl_{0,7} \to \mathbb{R}(4) \otimes \mathbb{R}(2) \oplus \mathbb{R}(4) \otimes \mathbb{R}(2)$

$$\Psi_{0,7}(\varepsilon_1) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\Psi_{0,7}(\varepsilon_2) = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$$\Psi_{0,7}(\varepsilon_3) = \left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

$$\Psi_{0,7}(\varepsilon_4) = \left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

$$\Psi_{0,7}(\varepsilon_5) = \left(\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$$

$$\Psi_{0,7}(\varepsilon_6)=\left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\otimes\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\otimes\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

$$\Psi_{0,7}(\varepsilon_7)=\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\otimes\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$$

Burada Kronecker çarpımı kullanılırsa aradığımız izomorfizm aşağıdaki gibi olur.

$Cl_{0,7}\rightarrow\mathbb{R}(8)\oplus\mathbb{R}(8)$

$$E_1=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$ olmak üzere $\Psi_{0,7}(\varepsilon_1)=(E_1,E_1)$

$$E_2=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$ olmak üzere $\Psi_{0,7}(\varepsilon_2)=(E_2,E_2)$

$$E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ olmak üzere } \Psi_{0,7}(\varepsilon_3) = (E_3, E_3)$$

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ olmak üzere } \Psi_{0,7}(\varepsilon_4) = (E_4, E_4)$$

$$E_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ olmak üzere } \Psi_{0,7}(\varepsilon_5) = (E_5, E_5)$$

$$E_6 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ olmak üzere } \Psi_{0,7}(\varepsilon_6) = (E_6, E_6)$$

$$E_7 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ olmak üzere } \Psi_{0,7}(\varepsilon_7) = (E_7, -E_7)$$

sonuçları elde edilir.

viii) $n = 8$ için $\Psi_{0,8} : Cl_{0,8} \to \mathbb{R}(16)$ izomorfizminin açık ifadesini, daha önce bulduğumuz izomorfizmlerin açık ifadelerini kullanarak bulalım.

$$
\begin{array}{llll}
Cl_{0,8} & \rightarrow & Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} & \rightarrow \mathbb{R}(2) \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{H} \\
\varepsilon_1 & \mapsto & 1 \otimes 1 \otimes 1 \otimes \varepsilon_1 & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes i \\
\varepsilon_2 & \mapsto & 1 \otimes 1 \otimes 1 \otimes \varepsilon_2 & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes j \\
\varepsilon_3 & \mapsto & 1 \otimes 1 \otimes e_1 \otimes \varepsilon_1\varepsilon_2 & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes 1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes k \\
\varepsilon_4 & \mapsto & 1 \otimes 1 \otimes e_2 \otimes \varepsilon_1\varepsilon_2 & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes k \\
\varepsilon_5 & \mapsto & 1 \otimes \varepsilon_1 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes i \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes k \\
\varepsilon_6 & \mapsto & 1 \otimes \varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes j \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes k \\
\varepsilon_7 & \mapsto & e_1 \otimes \varepsilon_1\varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 & \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes k \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes k \\
\varepsilon_8 & \mapsto & e_2 \otimes \varepsilon_1\varepsilon_2 \otimes e_1e_2 \otimes \varepsilon_1\varepsilon_2 & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes k \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes k
\end{array}
$$

$$
\begin{aligned}
Cl_{0,8} &\to \mathbb{R}(2)\otimes\mathbb{R}(2)\otimes\mathbb{H}\otimes\mathbb{H}\\
\varepsilon_1 &\mapsto \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes 1\otimes i\\
\varepsilon_2 &\mapsto \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes 1\otimes j\\
\varepsilon_3 &\mapsto \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix}\otimes 1\otimes k\\
\varepsilon_4 &\mapsto \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix}\otimes 1\otimes k\\
\varepsilon_5 &\mapsto \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}\otimes i\otimes k\\
\varepsilon_6 &\mapsto \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}\otimes j\otimes k\\
\varepsilon_7 &\mapsto \begin{bmatrix}0&1\\1&0\end{bmatrix}\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}\otimes k\otimes k\\
\varepsilon_8 &\mapsto \begin{bmatrix}1&0\\0&-1\end{bmatrix}\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}\otimes k\otimes k
\end{aligned}
$$

$$Cl_{0,8} \rightarrow \mathbb{R}(4) \otimes \mathbb{R}(4)$$

$$\varepsilon_1 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\varepsilon_2 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_3 \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_4 \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_5 \mapsto \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\varepsilon_6 \mapsto \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\varepsilon_7 \mapsto \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\varepsilon_8 \mapsto \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Burada yine Kronecker çarpımını kullanalım.

$Cl_{0,8} \longrightarrow \mathbb{R}(16)$

$$\varepsilon_1 \mapsto \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

$$
\varepsilon_2 \quad \mapsto \quad \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}
$$

$$
\varepsilon_3 \quad \mapsto \quad \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\varepsilon_4 \quad \mapsto \quad \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\varepsilon_5 \quad \mapsto \quad \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\varepsilon_6 \mapsto \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\varepsilon_7 \mapsto \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\varepsilon_8 \mapsto \begin{bmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

2) $m = n+8$ $(n \geq 1)$ iken $Cl_{0,m}$ ler için izomorfizmlerin elde edilmesi

$n \geq 1$ için $Cl_{0,n+8}$ Clifford cebirlerinin izomorf olduğu cebirleri bulmak mümkündür. $Cl_{0,n+8}$ cebrine iki kez dörtlü indirgeme formülü uygulanırsa

$$
\begin{aligned}
Cl_{0,n+8} &\cong Cl_{0,n+4} \otimes Cl_{2,0} \otimes Cl_{0,2} \cong Cl_{0,n} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \\
&\cong Cl_{0,n} \otimes Cl_{0,8}
\end{aligned}
$$

olur. Bunu daha açık olarak aşağıdaki şekilde ifade edelim.

$$
\begin{array}{llllll}
Cl_{0,n+8} & \rightarrow & Cl_{0,n+4}\otimes Cl_{2,0}\otimes Cl_{0,2} & \rightarrow & Cl_{0,n}\otimes Cl_{2,0}\otimes Cl_{0,2}\otimes Cl_{2,0}\otimes Cl_{0,2} \\
\varepsilon_1 & \mapsto & 1\otimes 1\otimes \varepsilon_1 & \mapsto & 1\otimes 1\otimes 1\otimes 1\otimes \varepsilon_1 \\
\varepsilon_2 & \mapsto & 1\otimes 1\otimes \varepsilon_2 & \mapsto & 1\otimes 1\otimes 1\otimes 1\otimes \varepsilon_2 \\
\varepsilon_3 & \mapsto & 1\otimes e_1\otimes \varepsilon_1\varepsilon_2 & \mapsto & 1\otimes 1\otimes 1\otimes e_1\otimes \varepsilon_1\varepsilon_2 \\
\varepsilon_4 & \mapsto & 1\otimes e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & 1\otimes 1\otimes 1\otimes e_2\otimes \varepsilon_1\varepsilon_2 \\
\varepsilon_5 & \mapsto & \varepsilon_1\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & 1\otimes 1\otimes \varepsilon_1\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 \\
\varepsilon_6 & \mapsto & \varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & 1\otimes 1\otimes \varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 \\
\varepsilon_7 & \mapsto & \varepsilon_3\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & 1\otimes e_1\otimes \varepsilon_1\varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 \\
\varepsilon_8 & \mapsto & \varepsilon_4\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & 1\otimes e_2\otimes \varepsilon_1\varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 \\
\varepsilon_9 & \mapsto & \varepsilon_5\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 \\
\varepsilon_{10} & \mapsto & \varepsilon_6\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & \varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 \\
\vdots & \mapsto & \vdots & \mapsto & \vdots \\
\varepsilon_{n+8} & \mapsto & \varepsilon_{n+4}\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2 & \mapsto & \varepsilon_n\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2\otimes e_1e_2\otimes \varepsilon_1\varepsilon_2
\end{array}
$$

Buradan da aşağıdaki sonuç elde edilir.

$$
\begin{array}{lll}
Cl_{0,n+8} & \rightarrow & Cl_{0,n}\otimes \mathbb{R}\,(16) \\
\varepsilon_1 & \mapsto & 1\otimes A_1 \\
\varepsilon_2 & \mapsto & 1\otimes A_2 \\
\varepsilon_3 & \mapsto & 1\otimes A_3 \\
\varepsilon_4 & \mapsto & 1\otimes A_4 \\
\varepsilon_5 & \mapsto & 1\otimes A_5 \\
\varepsilon_6 & \mapsto & 1\otimes A_6 \\
\varepsilon_7 & \mapsto & 1\otimes A_7 \\
\varepsilon_8 & \mapsto & 1\otimes A_8 \\
\varepsilon_9 & \mapsto & \varepsilon_1\otimes B \\
\varepsilon_{10} & \mapsto & \varepsilon_2\otimes B \\
\vdots & \mapsto & \vdots \\
\varepsilon_{n+8} & \mapsto & \varepsilon_n\otimes B
\end{array}
$$

Burada $A_1=\Psi_{0,8}\left(\varepsilon_1\right)$, $A_2=\Psi_{0,8}\left(\varepsilon_2\right)$, $A_3=\Psi_{0,8}\left(\varepsilon_3\right)$, $A_4=\Psi_{0,8}\left(\varepsilon_4\right)$, $A_5=\Psi_{0,8}\left(\varepsilon_5\right)$, $A_6=\Psi_{0,8}\left(\varepsilon_6\right)$, $A_7=\Psi_{0,8}\left(\varepsilon_7\right)$, $A_8=\Psi_{0,8}\left(\varepsilon_8\right)$ olmaktadır. Buradaki B matrisi ise aşağıdaki matristir.

$$B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

5 $Cl_{p,q}$ CLİFFORD CEBİRİNİN İZOMORFİZMLERİNİN ELDE EDİLMESİ

Önceki bölümdeki tabloda Clifford cebirlerinin izomorf olduğu matris cebirleri verilmiş oldu ancak $Cl_{0,n}$ tipindeki Clifford cebirleri dışında ilgili izomorfizmlerin açık ifadeleri verilmedi. Bu bölümdeki amacımız tabloda geçen tüm izomorfizmlerin açık ifadelerini elde etmek. Bundan böyle hem kısalık için hemde işlemlerde kolaylık sağlaması için

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_2, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \sigma_1\sigma_2$$
$$(\ \sigma_1^2 = \sigma_2^2 = I, \quad (\sigma_1\sigma_2)^2 = -I\ )$$

diyeceğiz ve $Cl_{0,n}$ lerde dahil olmak üzere tüm izomorfizmleri bu gösterimleri kullanarak elde edeceğiz.

1) $\mathbb{R}^1$ üzerindeki dejenere olmayan Clifford cebirlerinin (yani, $Cl_{1,0}$ ve $Cl_{0,1}$) izomorfizmlerinin açık ifadelerini önceki bölümde yazmıştık.

2) $\mathbb{R}^2$ üzerindeki dejenere olmayan Clifford cebirlerinin (yani, $Cl_{2,0}$;$Cl_{1,1}$ ve $Cl_{0,2}$) izomorfizmlerinin açık ifadelerini önceki bölümde yazmıştık.

3) $\mathbb{R}^3$ üzerindeki dejenere olmayan Clifford cebirleri tabloda görüldüğü gibi $Cl_{3,0} \cong \mathbb{C}(2)$, $Cl_{2,1} \cong \mathbb{R}(2) \oplus \mathbb{R}(2)$, $Cl_{1,2} \cong \mathbb{C}(2)$, $Cl_{0,3} \cong \mathbb{H} \oplus \mathbb{H}$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{3,0} : Cl_{3,0} \to \mathbb{C}(2)$

$$\begin{array}{ccccccc}
Cl_{3,0} & \to & Cl_{0,1} \otimes Cl_{2,0} & \to & \mathbb{C} \otimes \mathbb{R}(2) & \to & \mathbb{C}(2) \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & 1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \\
\varepsilon_2 & \mapsto & 1 \otimes \varepsilon_2 & \mapsto & 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_2 \\
\varepsilon_3 & \mapsto & e_1 \otimes \varepsilon_1\varepsilon_2 & \mapsto & i \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\sigma_1\sigma_2
\end{array}$$

ii) $\Psi_{2,1} : Cl_{2,1} \to \mathbb{R}(2) \oplus \mathbb{R}(2)$

$$\begin{array}{lllll}
Cl_{2,1} & \to & Cl_{1,0} \otimes Cl_{1,1} & \to & \mathbb{R}(2) \oplus \mathbb{R}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = (\sigma_1, \sigma_1) \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) = (\sigma_2, -\sigma_2) \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = (\sigma_1\sigma_2, \sigma_1\sigma_2)
\end{array}$$

iii) $\Psi_{1,2} : Cl_{1,2} \to \mathbb{C}(2)$

$$\begin{array}{lllllll}
Cl_{1,2} & \to & Cl_{0,1} \otimes Cl_{1,1} & \to & \mathbb{C} \otimes \mathbb{R}(2) & \to & \mathbb{C}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & 1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & 1 \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & i \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\sigma_2
\end{array}$$

iv) $\Psi_{0,3} : Cl_{0,3} \to \mathbb{H} \oplus \mathbb{H}$

$$\begin{array}{lllllll}
Cl_{0,3} & \to & Cl_{1,0} \otimes Cl_{0,2} & \to & (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} & \to & \mathbb{H} \oplus \mathbb{H} \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & (1,1) \otimes i & \mapsto & (i,i) \\
\varepsilon_2 & \mapsto & 1 \otimes \varepsilon_2 & \mapsto & (1,1) \otimes j & \mapsto & (j,j) \\
\varepsilon_3 & \mapsto & \varepsilon_1 \otimes \varepsilon_1\varepsilon_2 & \mapsto & (1,-1) \otimes k & \mapsto & (k,-k)
\end{array}$$

4) $\mathbb{R}^4$ üzerindeki dejenere olmayan Clifford cebirleri tabloda görüldüğü gibi $Cl_{4,0} \cong \mathbb{H}(2)$, $Cl_{3,1} \cong \mathbb{R}(2) \oplus \mathbb{R}(2)$, $Cl_{2,2} \cong \mathbb{C}(2)$, $Cl_{1,3} \cong \mathbb{H} \oplus \mathbb{H}$, $Cl_{0,4} \cong \mathbb{H}(2)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{4,0} : Cl_{4,0} \to \mathbb{H}(2)$

$$\begin{array}{lllllll}
Cl_{4,0} & \to & Cl_{0,2} \otimes Cl_{2,0} & \to & \mathbb{H} \otimes \mathbb{R}(2) & \to & \mathbb{H}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & 1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \\
e_2 & \mapsto & 1 \otimes e_2 & \mapsto & 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_2 \\
e_3 & \mapsto & e_1 \otimes e_1 e_2 & \mapsto & i \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\sigma_1\sigma_2 \\
e_4 & \mapsto & e_2 \otimes e_1 e_2 & \mapsto & j \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} = j\sigma_1\sigma_2
\end{array}$$

ii) $\Psi_{3,1} : Cl_{3,1} \to \mathbb{R}(4)$

$$\begin{array}{lllll}
Cl_{3,1} & \to & Cl_{2,0} \otimes Cl_{1,1} & \to & \mathbb{R}(2) \otimes \mathbb{R}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
e_2 & \mapsto & e_1 \otimes e_1 \varepsilon_1 & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
e_3 & \mapsto & e_2 \otimes e_1 \varepsilon_1 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{array}$$

$$\begin{array}{lll}
Cl_{3,1} & \to & \mathbb{R}(4) \\
e_1 & \mapsto & I \otimes \sigma_1 \\
e_2 & \mapsto & \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2
\end{array}$$

iii) $\Psi_{2,2} : Cl_{2,2} \to \mathbb{R}(4)$

$$\begin{array}{lllllll}
Cl_{2,2} & \to & Cl_{1,1} \otimes Cl_{1,1} & \to & \mathbb{R}(2) \otimes \mathbb{R}(2) & \to & \mathbb{R}(4) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto & I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2
\end{array}$$

iv) $\Psi_{1,3} : Cl_{1,3} \to \mathbb{H}(2)$

$$\begin{array}{lllllll}
Cl_{1,3} & \to & Cl_{0,2} \otimes Cl_{1,1} & \to & \mathbb{H} \otimes \mathbb{R}(2) & \to & \mathbb{H}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & 1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & 1 \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & i \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & j \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix} = j\sigma_2
\end{array}$$

v) $\Psi_{0,4} : Cl_{0,4} \to \mathbb{H}(2)$ (Bunu önceki bölümde yazmıştık, şimdi kısaltılmış şekliyle tekrar yazıyoruz.)

$$\begin{aligned}
\Psi_{0,4}(\varepsilon_1) &= iI \\
\Psi_{0,4}(\varepsilon_2) &= jI \\
\Psi_{0,4}(\varepsilon_3) &= k\sigma_1 \\
\Psi_{0,4}(\varepsilon_4) &= k\sigma_2
\end{aligned}$$

5) $\mathbb{R}^5$ üzerindeki dejenere olmayan Clifford cebirleri tabloda görüldüğü gibi $Cl_{5,0} \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$, $Cl_{4,1} \cong \mathbb{C}(4)$, $Cl_{3,2} \cong \mathbb{R}(4) \oplus \mathbb{R}(4)$, $Cl_{2,3} \cong \mathbb{C}(4)$, $Cl_{1,4} \cong \mathbb{H}(2) \oplus \mathbb{H}(2)$, $Cl_{0,5} \cong \mathbb{C}(4)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{5,0} : Cl_{5,0} \to \mathbb{H}(2) \oplus \mathbb{H}(2)$

$$\begin{array}{lllllll}
Cl_{5,0} & \to & Cl_{0,3} \otimes Cl_{2,0} & \to & (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{R}(2) & \to & \mathbb{H}(2) \oplus \mathbb{H}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & (1,1) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto & (\sigma_1, \sigma_1) \\
e_2 & \mapsto & 1 \otimes e_2 & \mapsto & (1,1) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & (\sigma_2, \sigma_2) \\
e_3 & \mapsto & \varepsilon_1 \otimes e_1 e_2 & \mapsto & (i,i) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & (i\sigma_1\sigma_2, i\sigma_1\sigma_2) \\
e_4 & \mapsto & \varepsilon_2 \otimes e_1 e_2 & \mapsto & (j,j) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & (j\sigma_1\sigma_2, j\sigma_1\sigma_2) \\
e_5 & \mapsto & \varepsilon_3 \otimes e_1 e_2 & \mapsto & (k,-k) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & (k\sigma_1\sigma_2, -k\sigma_1\sigma_2)
\end{array}$$

ii) $\Psi_{4,1} : Cl_{4,1} \to \mathbb{C}(4)$

$$\begin{array}{lll}
Cl_{4,1} & \to & \mathbb{C}(4) \\
e_1 & \mapsto & I \otimes \sigma_1 \\
e_2 & \mapsto & \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2
\end{array}$$

iii) $\Psi_{3,2} : Cl_{3,2} \to \mathbb{H}(2) \oplus \mathbb{H}(2)$

$$\begin{array}{lllll}
Cl_{3,2} & \to & Cl_{2,1} \otimes Cl_{1,1} & \to & (\mathbb{R}(2) \oplus \mathbb{R}(2)) \otimes \mathbb{R}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{array}$$

v) $\Psi_{1,4} : Cl_{1,4} \to \mathbb{H}(2) \oplus \mathbb{H}(2)$

$$\begin{array}{llllllll}
Cl_{1,4} & \to & Cl_{0,3} \otimes Cl_{1,1} & \to & (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{R}(2) & \to & \mathbb{H}(2) \oplus \mathbb{H}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & (1,1) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \mapsto & (\sigma_1, \sigma_1) \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & (1,1) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mapsto & (\sigma_1\sigma_2, \sigma_1\sigma_2) \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & (i,i) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & (i\sigma_2, i\sigma_2) \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & (j,j) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & (j\sigma_2, j\sigma_2) \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & (k,-k) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \mapsto & (k\sigma_2, -k\sigma_2)
\end{array}$$

vi) $\Psi_{0,5} : Cl_{0,5} \to \mathbb{C}(4)$ (Bunu önceki bölümde yazmıştık, şimdi kısaltılmış şekliyle tekrar yazıyoruz.)

$$\begin{aligned}
\Psi_{0,5}(\varepsilon_1) &= -i\sigma_2 \otimes I \\
\Psi_{0,5}(\varepsilon_2) &= -\sigma_1\sigma_2 \otimes I \\
\Psi_{0,5}(\varepsilon_3) &= -i\sigma_1 \otimes \sigma_1 \\
\Psi_{0,5}(\varepsilon_4) &= -i\sigma_1 \otimes \sigma_2 \\
\Psi_{0,5}(\varepsilon_5) &= \sigma_1 \otimes \sigma_1\sigma_2
\end{aligned}$$

6) $\mathbb{R}^6$ üzerindeki dejenere olmayan Clifford cebirleri tabloda görüldüğü gibi $Cl_{6,0} \cong \mathbb{H}(4)$, $Cl_{5,1} \cong \mathbb{H}(4)$, $Cl_{4,2} \cong \mathbb{R}(8)$, $Cl_{3,3} \cong \mathbb{R}(8)$, $Cl_{2,4} \cong \mathbb{H}(4)$, $Cl_{1,5} \cong \mathbb{H}(4)$, $Cl_{0,6} \cong \mathbb{R}(8)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{6,0} : Cl_{6,0} \to \mathbb{H}(4)$

$$\begin{array}{llllll}
Cl_{6,0} & \to & Cl_{0,4} \otimes Cl_{2,0} & \to & \mathbb{H}(2) \otimes \mathbb{R}(2) \cong \mathbb{H}(4) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes \sigma_1 \\
e_2 & \mapsto & 1 \otimes e_2 & \mapsto & I \otimes \sigma_2 \\
e_3 & \mapsto & \varepsilon_1 \otimes e_1e_2 & \mapsto & iI \otimes \sigma_1\sigma_2 \\
e_4 & \mapsto & \varepsilon_2 \otimes e_1e_2 & \mapsto & jI \otimes \sigma_1\sigma_2 \\
e_5 & \mapsto & \varepsilon_3 \otimes e_1e_2 & \mapsto & k\sigma_1 \otimes \sigma_1\sigma_2 \\
e_6 & \mapsto & \varepsilon_4 \otimes e_1e_2 & \mapsto & k\sigma_2 \otimes \sigma_1\sigma_2
\end{array}$$

ii) $\Psi_{5,1} : Cl_{5,1} \to \mathbb{H}(4)$

$$
\begin{array}{lllll}
Cl_{5,1} & \to & Cl_{4,0} \otimes Cl_{1,1} & \to & \mathbb{H}(2) \otimes \mathbb{R}(2) \cong \mathbb{H}(4) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & j\sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2
\end{array}
$$

iii) $\Psi_{4,2} : Cl_{4,2} \to \mathbb{R}(8)$

$$
\begin{array}{lllll}
Cl_{4,2} & \to & Cl_{3,1} \otimes Cl_{1,1} & \to & \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2
\end{array}
$$

iv) $\Psi_{3,3} : Cl_{3,3} \to \mathbb{R}(8)$

$$
\begin{array}{lllll}
Cl_{3,3} & \to & Cl_{2,2} \otimes Cl_{1,1} & \to & \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

iv) $\Psi_{2,4} : Cl_{2,4} \to \mathbb{H}(4)$

$$
\begin{array}{lllll}
Cl_{2,4} & \to & Cl_{1,3} \otimes Cl_{1,1} & \mapsto & \mathbb{H}(2) \otimes \mathbb{R}(2) \cong \mathbb{H}(4) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & j\sigma_2 \otimes \sigma_2
\end{array}
$$

v) $\Psi_{1,5} : Cl_{1,5} \to \mathbb{H}(4)$

$$\begin{array}{lllllll}
Cl_{1,5} & \to & Cl_{0,4}\otimes Cl_{1,1} & \to & \mathbb{H}(2)\otimes\mathbb{R}(2) & \to & \mathbb{H}(4) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix} & \mapsto & I\otimes\sigma_1 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix} & \mapsto & I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & \begin{bmatrix}i&0\\0&i\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix} & \mapsto & iI\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & \begin{bmatrix}j&0\\0&j\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix} & \mapsto & jI\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & \begin{bmatrix}0&k\\k&0\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix} & \mapsto & k\sigma_1\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & \begin{bmatrix}k&0\\0&-k\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix} & \mapsto & k\sigma_2\otimes\sigma_2
\end{array}$$

v) $\Psi_{0,6} : Cl_{0,6} \to \mathbb{R}(8)$ (Bunu önceki bölümde yazmıştık, şimdi kısaltılmış şekliyle tekrar yazıyoruz.)

$$\begin{aligned}
\Psi_{0,6}(\varepsilon_1) &= -\sigma_2\otimes\sigma_1\sigma_2\otimes I \\
\Psi_{0,6}(\varepsilon_2) &= -\sigma_1\sigma_2\otimes I\otimes I \\
\Psi_{0,6}(\varepsilon_3) &= -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1 \\
\Psi_{0,6}(\varepsilon_4) &= -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\Psi_{0,6}(\varepsilon_5) &= \sigma_1\otimes I\otimes\sigma_1\sigma_2 \\
\Psi_{0,6}(\varepsilon_6) &= -\sigma_2\otimes\sigma_1\otimes\sigma_1\sigma_2
\end{aligned}$$

7) $\mathbb{R}^7$ üzerindeki dejenere olmayan Clifford cebirleri tabloda görüldüğü gibi $Cl_{7,0} \cong \mathbb{C}(8)$, $Cl_{6,1} \cong \mathbb{H}(4)\oplus\mathbb{H}(4)$, $Cl_{5,2} \cong \mathbb{C}(8)$, $Cl_{4,3} \cong \mathbb{R}(8)\oplus\mathbb{R}(8)$, $Cl_{3,4} \cong \mathbb{C}(8)$, $Cl_{2,5} \cong \mathbb{H}(4)\oplus\mathbb{H}(4)$, $Cl_{1,6} \cong \mathbb{C}(8)$, $Cl_{0,7} \cong \mathbb{R}(8)\oplus\mathbb{R}(8)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{7,0} : Cl_{7,0} \to \mathbb{C}(8)$

$$\begin{array}{lllll}
Cl_{7,0} & \to & Cl_{0,5} \otimes Cl_{2,0} & \to & \mathbb{C}(4) \otimes \mathbb{R}(2) \cong \mathbb{C}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I_2 \otimes I_2 \otimes \sigma_1 \\
e_2 & \mapsto & 1 \otimes e_2 & \mapsto & I_2 \otimes I_2 \otimes \sigma_2 \\
e_3 & \mapsto & \varepsilon_1 \otimes e_1 e_2 & \mapsto & -\sigma_2 \otimes I_2 \otimes i\sigma_1\sigma_2 \\
e_4 & \mapsto & \varepsilon_2 \otimes e_1 e_2 & \mapsto & \sigma_1\sigma_2 \otimes I_2 \otimes \sigma_1\sigma_2 \\
e_5 & \mapsto & \varepsilon_3 \otimes e_1 e_2 & \mapsto & -\sigma_1 \otimes \sigma_1 \otimes i\sigma_1\sigma_2 \\
e_6 & \mapsto & \varepsilon_4 \otimes e_1 e_2 & \mapsto & -\sigma_1 \otimes \sigma_2 \otimes i\sigma_1\sigma_2 \\
e_7 & \mapsto & \varepsilon_5 \otimes e_1 e_2 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2
\end{array}$$

ii) $\Psi_{6,1} : Cl_{6,1} \to \mathbb{H}(4) \oplus \mathbb{H}(4)$

$$\begin{array}{lllll}
Cl_{6,1} & \to & Cl_{5,0} \otimes Cl_{1,1} & \to & (\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes \mathbb{R}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}, \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix}, \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\end{array}$$

$$\begin{array}{lll}
Cl_{6,1} & \to & \mathbb{H}(4) \oplus \mathbb{H}(4) \\
e_1 & \mapsto & (I \otimes \sigma_1, I \otimes \sigma_1) \\
e_2 & \mapsto & (\sigma_1 \otimes \sigma_2, \sigma_1 \otimes \sigma_2) \\
e_3 & \mapsto & (\sigma_2 \otimes \sigma_2, \sigma_2 \otimes \sigma_2) \\
e_4 & \mapsto & (i\sigma_1\sigma_2 \otimes \sigma_2, i\sigma_1\sigma_2 \otimes \sigma_2) \\
e_5 & \mapsto & (j\sigma_1\sigma_2 \otimes \sigma_2, j\sigma_1\sigma_2 \otimes \sigma_2)
\end{array}$$

$$\begin{array}{lll} e_6 & \mapsto & (k\sigma_1\sigma_2 \otimes \sigma_2, -k\sigma_1\sigma_2 \otimes \sigma_2) \\ \varepsilon_1 & \mapsto & (I \otimes \sigma_1\sigma_2, I \otimes \sigma_1\sigma_2) \end{array}$$

iii) $\Psi_{5,2} : Cl_{5,2} \to \mathbb{C}(8)$

$$\begin{array}{lllll} Cl_{5,2} & \to & Cl_{4,1} \otimes Cl_{1,1} & \to & \mathbb{C}(4) \otimes \mathbb{R}(2) \cong \mathbb{C}(8) \\ e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\ e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\ e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\ e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\ e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\ \varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\ \varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \end{array}$$

iv) $\Psi_{4,3} : Cl_{4,3} \to \mathbb{R}(8) \oplus \mathbb{R}(8)$

$$\begin{array}{lllll} Cl_{4,3} & \to & Cl_{3,2} \otimes Cl_{1,1} & \to & \mathbb{R}(8) \oplus \mathbb{R}(8) \\ e_1 & \mapsto & 1 \otimes e_1 & \mapsto & (I \otimes I \otimes \sigma_1, I \otimes I \otimes \sigma_1) \\ e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes \sigma_1 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2) \\ e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_1 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2) \\ e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_2 \otimes \sigma_2 \otimes \sigma_2, -\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\ \varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & (I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes \sigma_1\sigma_2) \\ \varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\ \varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \end{array}$$

v) $\Psi_{3,4} : Cl_{3,4} \to \mathbb{C}(8)$

$$\begin{array}{lllll} Cl_{3,4} & \to & \mathbb{C}(4) \otimes \mathbb{R}(2) & \to & \mathbb{C}(4) \otimes \mathbb{R}(2) \cong \mathbb{C}(8) \\ e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\ e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\ e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\ \varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\ \varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\ \varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\ \varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \end{array}$$

vi) $\Psi_{2,5} : Cl_{2,5} \to \mathbb{H}(4) \oplus \mathbb{H}(4)$

$$\begin{array}{lllll}
Cl_{2,5} & \to & Cl_{1,4} \otimes Cl_{1,1} & \to & (\mathbb{H}(2) \oplus \mathbb{H}(2)) \otimes \mathbb{R}(2) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & \left(\begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}, \begin{bmatrix} -k & 0 \\ 0 & k \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\end{array}$$

$$\begin{array}{lll}
Cl_{2,5} & \to & \mathbb{H}(4) \oplus \mathbb{H}(4) \\
e_1 & \mapsto & (I \otimes \sigma_1, I \otimes \sigma_1) \\
e_2 & \mapsto & (\sigma_1 \otimes \sigma_2, \sigma_1 \otimes \sigma_2) \\
\varepsilon_1 & \mapsto & (I \otimes \sigma_1\sigma_2, I \otimes \sigma_1\sigma_2) \\
\varepsilon_2 & \mapsto & (\sigma_1\sigma_2 \otimes \sigma_2, \sigma_1\sigma_2 \otimes \sigma_2) \\
\varepsilon_3 & \mapsto & (i\sigma_2 \otimes \sigma_2, i\sigma_2 \otimes \sigma_2) \\
\varepsilon_4 & \mapsto & (j\sigma_2 \otimes \sigma_2, j\sigma_2 \otimes \sigma_2) \\
\varepsilon_5 & \mapsto & (k\sigma_2 \otimes \sigma_2, -k\sigma_2 \otimes \sigma_2)
\end{array}$$

vii) $\Psi_{1,6} : Cl_{1,6} \to \mathbb{C}(8)$

$$\begin{array}{lllll}
Cl_{1,6} & \to & Cl_{0,5} \otimes Cl_{1,1} & \to & \mathbb{C}(4) \otimes \mathbb{R}(2) \cong \mathbb{C}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_2 \otimes I \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes \sigma_2
\end{array}$$

$$\begin{array}{lll}
\varepsilon_4 & \mapsto \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto -i\sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto -i\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2
\end{array}$$

viii) $\Psi_{0,7} : Cl_{0,7} \to \mathbb{H}(4) \oplus \mathbb{H}(4)$ (Bunu önceki bölümde yazmıştık, şimdi kısaltılmış şekliyle tekrar yazıyoruz.)

$$\begin{array}{ll}
Cl_{0,7} & \to \mathbb{R}(8) \oplus \mathbb{R}(8) \\
\varepsilon_1 & \mapsto (-\sigma_2 \otimes \sigma_1\sigma_2 \otimes I, -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I) \\
\varepsilon_2 & \mapsto (-\sigma_1\sigma_2 \otimes I \otimes I, -\sigma_1\sigma_2 \otimes I \otimes I) \\
\varepsilon_3 & \mapsto (-\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1, -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1) \\
\varepsilon_4 & \mapsto (-\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2, -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\varepsilon_5 & \mapsto (\sigma_1 \otimes I \otimes \sigma_1\sigma_2, \sigma_1 \otimes I \otimes \sigma_1\sigma_2) \\
\varepsilon_6 & \mapsto (-\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2, -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2) \\
\varepsilon_7 & \mapsto (\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2, \sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2)
\end{array}$$

8) $\mathbb{R}^8$ üzerindeki dejenere olmayan Clifford cebirleri tabloda görüldüğü gibi $Cl_{8,0} \cong \mathbb{R}(16)$, $Cl_{7,1} \cong \mathbb{H}(8)$, $Cl_{6,2} \cong \mathbb{H}(8)$, $Cl_{5,3} \cong \mathbb{R}(16)$, $Cl_{4,4} \cong \mathbb{R}(16)$, $Cl_{3,5} \cong \mathbb{H}(8)$, $Cl_{2,6} \cong \mathbb{H}(8)$, $Cl_{1,7} \cong \mathbb{R}(16)$, $Cl_{0,8} \cong \mathbb{R}(16)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,0} : Cl_{8,0} \to \mathbb{R}(16)$

$$\begin{array}{ll}
Cl_{8,0} & \to \mathbb{R}(16) \\
e_1 & \mapsto I_2 \otimes I_2 \otimes I_2 \otimes \sigma_1 \\
e_2 & \mapsto I_2 \otimes I_2 \otimes I_2 \otimes \sigma_2 \\
e_3 & \mapsto -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I_2 \otimes \sigma_1\sigma_2 \\
e_4 & \mapsto -\sigma_1\sigma_2 \otimes I_2 \otimes I_2 \otimes \sigma_1\sigma_2 \\
e_5 & \mapsto -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \\
e_6 & \mapsto -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \\
e_7 & \mapsto \sigma_1 \otimes I_2 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_8 & \mapsto -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2
\end{array}$$

ii) $\Psi_{7,1} : Cl_{7,1} \to \mathbb{H}(8)$

$$\begin{array}{lllll}
Cl_{7,1} & \to & Cl_{6,0} \otimes Cl_{1,1} & \to & \mathbb{H}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & iI \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & jI \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2
\end{array}$$

iii) $\Psi_{6,2} : Cl_{6,2} \to \mathbb{H}(8)$

$$\begin{array}{lllll}
Cl_{6,2} & \to & Cl_{5,1} \otimes Cl_{1,1} & \to & \mathbb{H}(4) \otimes \mathbb{R}(2) \cong \mathbb{H}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & j\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2
\end{array}$$

iv) $\Psi_{5,3} : Cl_{5,3} \to \mathbb{R}(16)$

$$\begin{array}{lllll}
Cl_{5,3} & \to & Cl_{4,2} \otimes Cl_{1,1} & \to & \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

v) $\Psi_{4,4} : Cl_{4,4} \to \mathbb{R}(16)$

$$\begin{array}{lllll}
Cl_{4,4} & \to & Cl_{3,3} \otimes Cl_{1,1} & \to & \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

vi) $\Psi_{3,5} : Cl_{3,5} \to \mathbb{H}(8)$

$$\begin{array}{lll}
Cl_{3,5} & \to & \mathbb{H}(8) \\
e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & j\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

vii) $\Psi_{2,6} : Cl_{2,6} \to \mathbb{H}(8)$

$$\begin{array}{lllll}
Cl_{2,6} & \to & Cl_{1,5} \otimes Cl_{1,1} & \to & \mathbb{H}(4) \otimes \mathbb{R}(2) \cong \mathbb{H}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & iI \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & jI \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

viii) $\Psi_{1,7} : Cl_{1,7} \to \mathbb{R}(16)$

$$\begin{array}{lllll}
Cl_{1,7} & \to & Cl_{0,6} \otimes Cl_{1,1} & \to & \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2
\end{array}$$

ix) $\Psi_{0,8} : Cl_{0,8} \to \mathbb{R}(16)$ (Bunu önceki bölümde yazmıştık, şimdi kısaltılmış şekliyle tekrar yazıyoruz.)

$$\begin{array}{lll}
Cl_{0,8} & \to & \mathbb{R}(16) \\
\varepsilon_1 & \mapsto & -I \otimes I \otimes \sigma_2 \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & -I \otimes I \otimes \sigma_1\sigma_2 \otimes I \\
\varepsilon_3 & \mapsto & -I \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \\
\varepsilon_4 & \mapsto & -I \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \\
\varepsilon_5 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes I \\
\varepsilon_6 & \mapsto & -I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1 \\
\varepsilon_7 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

9) $\mathbb{R}^9$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,1} \cong \mathbb{C}(16)$, $Cl_{7,2} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$, $Cl_{6,3} \cong \mathbb{C}(16)$, $Cl_{5,4} \cong \mathbb{R}(16) \oplus \mathbb{R}(16)$, $Cl_{4,5} \cong \mathbb{C}(16)$, $Cl_{3,6} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$, $Cl_{2,7} \cong \mathbb{C}(16)$, $Cl_{1,8} \cong \mathbb{R}(16) \oplus \mathbb{R}(16)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

ii) $\Psi_{8,1} : Cl_{8,1} \to \mathbb{C}(16)$

$$
\begin{array}{llllll}
Cl_{8,1} & \to & Cl_{7,0} \otimes Cl_{1,1} & \to & \mathbb{C}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_1 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_1 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & e_7 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2
\end{array}
$$

ii) $\Psi_{7,2} : Cl_{7,2} \to \mathbb{H}(8) \oplus \mathbb{H}(8)$

$$
\begin{array}{llllll}
Cl_{7,2} & \to & Cl_{6,1} \otimes Cl_{1,1} & \to & \mathbb{H}(8) \oplus \mathbb{H}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & (I \otimes I \otimes \sigma_1, I \otimes I \otimes \sigma_1) \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes \sigma_1 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2) \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_1 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2) \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & (i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & (j\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, j\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & (k\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, -k\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & (I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes \sigma_1\sigma_2) \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes \sigma_1\sigma_2 \otimes \sigma_2)
\end{array}
$$

iii) $\Psi_{6,3} : Cl_{6,3} \to \mathbb{C}\,(16)$

$$\begin{array}{llllll}
Cl_{6,3} & \to & Cl_{5,2} \otimes Cl_{1,1} & \to & \mathbb{C}\,(8) \otimes \mathbb{R}\,(2) \cong \mathbb{C}\,(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

iv) $\Psi_{5,4} : Cl_{5,4} \to \mathbb{R}\,(16) \oplus \mathbb{R}\,(16)$

$$\begin{array}{llllll}
Cl_{5,4} & \to & Cl_{4,3} \otimes Cl_{1,1} & \to & \mathbb{R}\,(16) \oplus \mathbb{R}\,(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & (I \otimes I \otimes I \otimes \sigma_1, I \otimes I \otimes I \otimes \sigma_1) \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes I \otimes \sigma_1 \otimes \sigma_2, I \otimes I \otimes \sigma_1 \otimes \sigma_2) \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2) \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & (I \otimes I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes I \otimes \sigma_1\sigma_2) \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & (I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & (\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)
\end{array}$$

v) $\Psi_{4,5} : Cl_{4,5} \rightarrow \mathbb{C}(16)$

$$\begin{array}{lllll}
Cl_{4,5} & \rightarrow & \mathbb{C}(8) \otimes \mathbb{R}(2) & \rightarrow & \mathbb{C}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

vi) $\Psi_{3,6} : Cl_{3,6} \rightarrow \mathbb{H}(8) \oplus \mathbb{H}(8)$

$$\begin{array}{ll}
\Psi_{3,6}(e_1) = & (I \otimes I \otimes \sigma_1, I \otimes I \otimes \sigma_1) \\
\Psi_{3,6}(e_2) = & (I \otimes \sigma_1 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2) \\
\Psi_{3,6}(e_3) = & (\sigma_1 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{3,6}(\varepsilon_1) = & (I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes \sigma_1\sigma_2) \\
\Psi_{3,6}(\varepsilon_2) = & (I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\Psi_{3,6}(\varepsilon_3) = & (\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{3,6}(\varepsilon_4) = & (i\sigma_2 \otimes \sigma_2 \otimes \sigma_2, i\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{3,6}(\varepsilon_5) = & (j\sigma_2 \otimes \sigma_2 \otimes \sigma_2, j\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{3,6}(\varepsilon_6) = & (k\sigma_2 \otimes \sigma_2 \otimes \sigma_2, -k\sigma_2 \otimes \sigma_2 \otimes \sigma_2)
\end{array}$$

vii) $\Psi_{2,7} : Cl_{2,7} \rightarrow \mathbb{C}(16)$

$$\begin{array}{lllll}
Cl_{2,7} & \rightarrow & Cl_{1,6} \otimes Cl_{1,1} & \rightarrow & \mathbb{C}(8) \otimes \mathbb{R}(2) \cong \mathbb{C}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2
\end{array}$$

$$\begin{array}{lllll}
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_2 \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

viii) $\Psi_{1,8} : Cl_{1,8} \to \mathbb{R}(16) \oplus \mathbb{R}(16)$

$$\begin{array}{lll}
\Psi_{1,8}(e_1) & = & (I \otimes I \otimes I \otimes \sigma_1, I \otimes I \otimes I \otimes \sigma_1) \\
\Psi_{1,8}(\varepsilon_1) & = & (I \otimes I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes I \otimes \sigma_1\sigma_2) \\
\Psi_{1,8}(\varepsilon_2) & = & (-\sigma_2 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_2, -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_2) \\
\Psi_{1,8}(\varepsilon_3) & = & (-\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_2, -\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_2) \\
\Psi_{1,8}(\varepsilon_4) & = & (-\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2, -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2) \\
\Psi_{1,8}(\varepsilon_5) & = & (-\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{1,8}(\varepsilon_6) & = & (\sigma_1 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2, \sigma_1 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\Psi_{1,8}(\varepsilon_7) & = & (-\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2, -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\Psi_{1,8}(\varepsilon_8) & = & (\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2, \sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2)
\end{array}$$

10) $\mathbb{R}^{10}$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,2} \cong \mathbb{H}(16)$, $Cl_{7,3} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$, $Cl_{6,4} \cong \mathbb{C}(16)$, $Cl_{5,5} \cong \mathbb{R}(16) \oplus \mathbb{R}(16)$, $Cl_{4,6} \cong \mathbb{C}(16)$, $Cl_{3,7} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$, $Cl_{2,8} \cong \mathbb{C}(16)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,2} : Cl_{8,2} \to \mathbb{H}(16)$

$$\begin{array}{lllll}
Cl_{8,2} & \to & Cl_{7,1} \otimes Cl_{1,1} & \to & \mathbb{H}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & iI \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & jI \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & e_7 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2
\end{array}$$

ii) $\Psi_{7,3} : Cl_{7,3} \to \mathbb{H}(16)$

$$
\begin{array}{lllll}
Cl_{7,3} & \to & Cl_{6,2} \otimes Cl_{1,1} & \to & \mathbb{H}(8) \otimes \mathbb{R}(2) \cong \mathbb{H}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & j\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

iii) $\Psi_{6,4} : Cl_{6,4} \to \mathbb{R}(32)$

$$
\begin{array}{lllll}
Cl_{6,4} & \to & Cl_{5,3} \otimes Cl_{1,1} & \to & \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

iv) $\Psi_{5,5} : Cl_{5,5} \to \mathbb{R}(32)$

$$\begin{array}{lllll}
Cl_{5,5} & \to & Cl_{4,4} \otimes Cl_{1,1} & \to & \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

v) $\Psi_{4,6} : Cl_{4,6} \to \mathbb{H}(16)$

$$\begin{array}{lllll}
Cl_{4,6} & \to & \mathbb{H}(8) \otimes \mathbb{R}(2) & \to & \mathbb{H}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & j\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

vi) $\Psi_{3,7} : Cl_{3,7} \to \mathbb{H}(16)$

$$\begin{array}{lllll}
Cl_{3,7} & \to & Cl_{2,6} \otimes Cl_{1,1} & \to & \mathbb{H}(8) \otimes \mathbb{R}(2) \cong \mathbb{H}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2
\end{array}$$

$$
\begin{array}{lllll}
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & iI \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & jI \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

vii) $\Psi_{2,8} : Cl_{2,8} \to \mathbb{R}(32)$

$$
\begin{array}{lllll}
Cl_{2,8} & \to & Cl_{1,7} \otimes Cl_{1,1} & \to & \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

11) $\mathbb{R}^{11}$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,3} \cong \mathbb{C}(16)$, $Cl_{7,4} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$, $Cl_{6,5} \cong \mathbb{C}(16)$, $Cl_{5,6} \cong \mathbb{R}(16) \oplus \mathbb{R}(16)$, $Cl_{4,7} \cong \mathbb{C}(16)$, $Cl_{3,8} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,3} : Cl_{8,3} \to \mathbb{H}(16) \oplus \mathbb{H}(16)$

$$
\begin{array}{lll}
\Psi_{8,3}(e_1) & = & (I \otimes I \otimes I \otimes \sigma_1, I \otimes I \otimes I \otimes \sigma_1) \\
\Psi_{8,3}(e_2) & = & (I \otimes I \otimes \sigma_1 \otimes \sigma_2, I \otimes I \otimes \sigma_1 \otimes \sigma_2) \\
\Psi_{8,3}(e_3) & = & (I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{8,3}(e_4) & = & (\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{8,3}(e_5) & = & (\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{8,3}(e_6) & = & (i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)
\end{array}
$$

$$
\begin{aligned}
\Psi_{8,3}(e_7) &= (j\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, j\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{8,3}(e_8) &= (k\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, -k\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{8,3}(\varepsilon_1) &= (I \otimes I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes I \otimes \sigma_1\sigma_2) \\
\Psi_{8,3}(\varepsilon_2) &= (I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\Psi_{8,3}(\varepsilon_3) &= (I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2)
\end{aligned}
$$

ii) $\Psi_{7,4} : Cl_{7,4} \to \mathbb{C}(32)$

$$
\begin{array}{lclcl}
Cl_{7,4} & \to & Cl_{6,3} \otimes Cl_{1,1} & \to & \mathbb{C}(16) \otimes \mathbb{R}(2) \cong \mathbb{C}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

iii) $\Psi_{6,5} : Cl_{6,5} \to \mathbb{R}(32) \oplus \mathbb{R}(32)$

$$
\begin{aligned}
\Psi_{6,5}(e_1) &= (I \otimes I \otimes I \otimes I \otimes \sigma_1, I \otimes I \otimes I \otimes I \otimes \sigma_1) \\
\Psi_{6,5}(e_2) &= (I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2, I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2) \\
\Psi_{6,5}(e_3) &= (I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{6,5}(e_4) &= (I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{6,5}(e_5) &= (\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{6,5}(e_6) &= (\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{6,5}(\varepsilon_1) &= (I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2) \\
\Psi_{6,5}(\varepsilon_2) &= (I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\Psi_{6,5}(\varepsilon_3) &= (I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{6,5}(\varepsilon_4) &= (I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{6,5}(\varepsilon_5) &= (\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)
\end{aligned}
$$

iv) $\Psi_{5,6} : Cl_{5,6} \to \mathbb{C}(32)$

$$\begin{array}{lllll}
Cl_{5,6} & \to & \mathbb{C}(16)\otimes\mathbb{R}(2) & \to & \mathbb{C}(32) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes \sigma_1 \\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes \sigma_1\otimes \sigma_2 \\
e_3 & \mapsto & e_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes \sigma_1\otimes \sigma_2\otimes \sigma_2 \\
e_4 & \mapsto & e_3\otimes e_1\varepsilon_1 & \mapsto & I\otimes \sigma_1\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2 \\
e_5 & \mapsto & e_4\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1\otimes \varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes \sigma_1\sigma_2\otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes \sigma_1\sigma_2\otimes \sigma_2\otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & I\otimes \sigma_1\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & i\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2
\end{array}$$

v) $\Psi_{4,7} : Cl_{4,7} \to \mathbb{H}(16)\oplus\mathbb{H}(16)$

$$\begin{array}{ll}
\Psi_{4,7}(e_1) = & (I\otimes I\otimes I\otimes \sigma_1, I\otimes I\otimes I\otimes \sigma_1) \\
\Psi_{4,7}(e_2) = & (I\otimes I\otimes \sigma_1\otimes \sigma_2, I\otimes I\otimes \sigma_1\otimes \sigma_2) \\
\Psi_{4,7}(e_3) = & (I\otimes \sigma_1\otimes \sigma_2\otimes \sigma_2, I\otimes \sigma_1\otimes \sigma_2\otimes \sigma_2) \\
\Psi_{4,7}(e_4) = & (\sigma_1\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2, \sigma_1\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2) \\
\Psi_{4,7}(\varepsilon_1) = & (I\otimes I\otimes I\otimes \sigma_1\sigma_2, I\otimes I\otimes I\otimes \sigma_1\sigma_2) \\
\Psi_{4,7}(\varepsilon_2) = & (I\otimes I\otimes \sigma_1\sigma_2\otimes \sigma_2, I\otimes I\otimes \sigma_1\sigma_2\otimes \sigma_2) \\
\Psi_{4,7}(\varepsilon_3) = & (I\otimes \sigma_1\sigma_2\otimes \sigma_2\otimes \sigma_2, I\otimes \sigma_1\sigma_2\otimes \sigma_2\otimes \sigma_2) \\
\Psi_{4,7}(\varepsilon_4) = & (\sigma_1\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2, \sigma_1\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2) \\
\Psi_{4,7}(\varepsilon_5) = & (i\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2, i\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2) \\
\Psi_{4,7}(\varepsilon_6) = & (j\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2, j\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2) \\
\Psi_{4,7}(\varepsilon_7) = & (k\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2, -k\sigma_2\otimes \sigma_2\otimes \sigma_2\otimes \sigma_2)
\end{array}$$

vi) $\Psi_{3,8} : Cl_{3,8} \to \mathbb{C}(32)$

$$\begin{array}{lllll}
Cl_{3,8} & \to & Cl_{2,7}\otimes Cl_{1,1} & \to & \mathbb{C}(16)\otimes\mathbb{R}(2)\cong\mathbb{C}(32) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes \sigma_1 \\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes \sigma_1\otimes \sigma_2 \\
e_3 & \mapsto & e_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes \sigma_1\otimes \sigma_2\otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I\otimes \varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes \sigma_1\sigma_2
\end{array}$$

$$
\begin{array}{lllll}
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_2 \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_1 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & -i\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

12) $\mathbb{R}^{12}$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,4} \cong \mathbb{C}(16)$, $Cl_{7,4} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$, $Cl_{6,5} \cong \mathbb{C}(16)$, $Cl_{5,6} \cong \mathbb{R}(16) \oplus \mathbb{R}(16)$, $Cl_{4,7} \cong \mathbb{C}(16)$, $Cl_{3,8} \cong \mathbb{H}(8) \oplus \mathbb{H}(8)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,4} : Cl_{8,4} \to \mathbb{H}(32)$

$$
\begin{array}{lllll}
Cl_{8,4} & \to & Cl_{7,3} \otimes Cl_{1,1} & \to & \mathbb{H}(16) \otimes \mathbb{R}(2) \cong \mathbb{H}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & e_7 \otimes e_1\varepsilon_1 & \mapsto & j\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

ii) $\Psi_{7,5} : Cl_{7,5} \to \mathbb{R}(64)$

$$\begin{array}{lclcl}
Cl_{7,5} & \to & Cl_{6,4} \otimes Cl_{1,1} & \to & \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

ii) $\Psi_{6,6} : Cl_{6,6} \to \mathbb{R}(64)$

$$\begin{array}{lclcl}
Cl_{6,6} & \to & Cl_{5,5} \otimes Cl_{1,1} & \to & \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

iii) $\Psi_{5,7} : Cl_{5,7} \to \mathbb{H}(32)$

$$\begin{array}{lllll}
Cl_{5,7} & \to & \mathbb{H}(16) \otimes \mathbb{R}(2) & \to & \mathbb{H}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & i\,\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & j\,\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

iv) $\Psi_{4,8} : Cl_{4,8} \to \mathbb{H}(32)$

$$\begin{array}{lllll}
Cl_{4,8} & \to & Cl_{3,7} \otimes Cl_{1,1} & \to & \mathbb{H}(16) \otimes \mathbb{R}(2) \cong \mathbb{H}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & iI \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & jI \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto & k\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

13) $\mathbb{R}^{13}$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,5} \cong \mathbb{C}(64)$, $Cl_{7,6} \cong \mathbb{R}(64) \oplus \mathbb{R}(64)$, $Cl_{6,7} \cong \mathbb{C}(64)$, $Cl_{5,8} \cong \mathbb{H}(32) \oplus \mathbb{H}(32)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,5} : Cl_{8,5} \to \mathbb{C}(64)$

$$
\begin{array}{lclcl}
Cl_{8,5} & \to & Cl_{7,4} \otimes Cl_{1,1} & \to & \mathbb{C}(32) \otimes \mathbb{R}(2) \cong \mathbb{C}(64) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & e_7 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

ii) $\Psi_{7,6} : Cl_{7,6} \to \mathbb{R}(64) \oplus \mathbb{R}(64)$

$$\Psi_{7,6}(e_1) = (I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1, I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1)$$

$$\Psi_{7,6}(e_2) = (I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2, I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2)$$

$$\Psi_{7,6}(e_3) = (I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(e_4) = (I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(e_5) = (I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(e_6) = (\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(e_7) = (\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(\varepsilon_1) = (I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2)$$

$$\Psi_{7,6}(\varepsilon_2) = (I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(\varepsilon_3) = (I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(\varepsilon_4) = (I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(\varepsilon_5) = (I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

$$\Psi_{7,6}(\varepsilon_6) = (\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)$$

iii) $\Psi_{6,7} : Cl_{6,7} \to \mathbb{C}(64)$

$$
\begin{array}{lllll}
Cl_{6,7} & \to & \mathbb{C}(32) \otimes \mathbb{R}(2) & \to & \mathbb{C}(64) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

iv) $\Psi_{5,8} : Cl_{5,8} \to \mathbb{H}(32) \oplus \mathbb{H}(32)$

$$
\begin{array}{ll}
\Psi_{5,8}(e_1) = & (I \otimes I \otimes I \otimes I \otimes \sigma_1, I \otimes I \otimes I \otimes I \otimes \sigma_1) \\
\Psi_{5,8}(e_2) = & (I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2, I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2) \\
\Psi_{5,8}(e_3) = & (I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(e_4) = & (I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(e_5) = & (\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(\varepsilon_1) = & (I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2, I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2) \\
\Psi_{5,8}(\varepsilon_2) = & (I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2, I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(\varepsilon_3) = & (I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(\varepsilon_4) = & (I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(\varepsilon_5) = & (\sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(\varepsilon_6) = & (i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(\varepsilon_7) = & (j\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, j\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2) \\
\Psi_{5,8}(\varepsilon_8) = & (k\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2, -k\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2)
\end{array}
$$

14) $\mathbb{R}^{14}$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,6} \cong \mathbb{R}(128)$, $Cl_{7,7} \cong \mathbb{R}(128)$, $Cl_{6,8} \cong \mathbb{H}(64)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,6} : Cl_{8,6} \to \mathbb{C}(64)$

$Cl_{8,6}$	$\to$	$Cl_{7,5} \otimes Cl_{1,1}$	$\to$	$\mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128)$
e_1	$\mapsto$	$1 \otimes e_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1$
e_2	$\mapsto$	$e_1 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2$
e_3	$\mapsto$	$e_2 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2$
e_4	$\mapsto$	$e_3 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_5	$\mapsto$	$e_4 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_6	$\mapsto$	$e_5 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_7	$\mapsto$	$e_6 \otimes e_1\varepsilon_1$	$\mapsto$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_8	$\mapsto$	$e_7 \otimes e_1\varepsilon_1$	$\mapsto$	$\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
ε_1	$\mapsto$	$1 \otimes \varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2$
ε_2	$\mapsto$	$\varepsilon_1 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2$
ε_3	$\mapsto$	$\varepsilon_2 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2$
ε_4	$\mapsto$	$\varepsilon_3 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
ε_5	$\mapsto$	$\varepsilon_4 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
ε_6	$\mapsto$	$\varepsilon_5 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$

ii) $\Psi_{7,7} : Cl_{7,7} \to \mathbb{R}(128)$

$Cl_{7,7}$	$\to$	$Cl_{6,6} \otimes Cl_{1,1}$	$\to$	$\mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128)$
e_1	$\mapsto$	$1 \otimes e_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1$
e_2	$\mapsto$	$e_1 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2$
e_3	$\mapsto$	$e_2 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_4	$\mapsto$	$e_3 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_5	$\mapsto$	$e_4 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_6	$\mapsto$	$e_5 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
e_7	$\mapsto$	$e_6 \otimes e_1\varepsilon_1$	$\mapsto$	$\sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$
ε_1	$\mapsto$	$1 \otimes \varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2$
ε_2	$\mapsto$	$\varepsilon_1 \otimes e_1\varepsilon_1$	$\mapsto$	$I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2$

$$\begin{array}{lllll}
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

ii) $\Psi_{6,8} : Cl_{6,8} \to \mathbb{H}(64)$

$$\begin{array}{lllll}
Cl_{6,8} & \to & \mathbb{H}(32) \otimes \mathbb{R}(2) & \to & \mathbb{H}(64) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & i\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto & j\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 .
\end{array}$$

15) $\mathbb{R}^{15}$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,7} \cong \mathbb{R}(128)$, $Cl_{7,8} \cong \mathbb{R}(128)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,7} : Cl_{8,7} \to \mathbb{R}(128) \oplus \mathbb{R}(128)$

$B_1 = I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1$ olmak üzere,

$\Psi_{8,7}(e_1) = (B_1, B_1)$ dir.

$B_2 = I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(e_2) = (B_2, B_2)$ dir.

$B_3 = I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2$

$\Psi_{8,7}(e_3) = (B_3, B_3)$ dir.

$B_4 = I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(e_4) = (B_4, B_4)$ dir.

$B_5 = I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(e_5) = (B_5, B_5)$ dir.

$B_6 = I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(e_6) = (B_6, B_6)$ dir.

$B_7 = \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(e_7) = (B_7, B_7)$ dir.

$B_8 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(e_8) = (B_8, -B_8)$ dir.

$B_9 = I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2$ olmak üzere,

$\Psi_{8,7}(\varepsilon_1) = (B_9, B_9)$ dir.

$B_{10} = I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(\varepsilon_2) = (B_{10}, B_{10})$ dir.

$B_{11} = I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(\varepsilon_3) = (B_{11}, B_{11})$ dir.

$B_{12} = I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(\varepsilon_4) = (B_{12}, B_{12})$ dir.

$B_{13} = I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(\varepsilon_5) = (B_{13}, B_{13})$ dir.

$B_{14} = I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(\varepsilon_6) = (B_{14}, B_{14})$ dir.

$B_{15} = \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2$ olmak üzere,

$\Psi_{8,7}(\varepsilon_7) = (B_{15}, B_{15})$ dir.

ii) $\Psi_{7,8} : Cl_{7,8} \to \mathbb{C}\,(128)$

$$
\begin{array}{llll}
Cl_{7,8} & \to & \mathbb{C}\,(64) \otimes \mathbb{R}\,(2) & \to \quad \mathbb{C}\,(128) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto \quad I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto \quad \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto \quad I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto \quad I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto \quad \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto \quad i\,\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

16) $\mathbb{R}^{16}$ üzerindeki dejenere olmayan Clifford cebirlerinden tabloda görülenler $Cl_{8,8} \cong \mathbb{R}\,(128)$ şeklindedir. Bu izomorfizmlerin açık ifadeleri aşağıdaki gibidir.

i) $\Psi_{8,8} : Cl_{8,8} \to \mathbb{R}\,(128)$

$$
\begin{array}{lclcl}
Cl_{8,8} & \to & Cl_{7,7} \otimes Cl_{1,1} & \to & \mathbb{R}\,(64) \otimes \mathbb{R}\,(2) \cong \mathbb{R}\,(128) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & e_7 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

6 MATRİS CEBİRLERİNİN TEMSİLLERİ

Önceki bölümde tüm reel dejenere olmayan Clifford cebirlerinin $\mathbb{R}(n), \mathbb{C}(n)$, $\mathbb{H}(n)$ yada $\mathbb{R}(n) \oplus \mathbb{R}(n), \mathbb{H}(n) \oplus \mathbb{H}(n)$ tipindeki matris cebirlerine izomorf olduğunu gördük. Bu nedenle Clifford cebirlerinin temsillerini inceleyebilmek için öncelikle bu tip matris cebirlerinin temsillerini anlamak gerekmektedir. Bu bölümde öncelikle matris cebirlerinin temsillerini inceleyeceğiz. $\mathbb{C}(n)$ cebrini reel cebir veya kompleks cebir, $\mathbb{R}(n)$ ve $\mathbb{H}(n)$ cebirlerini de reel cebir olarak alacağız. F ile, $\mathbb{R}, \mathbb{C}$ veya $\mathbb{H}$ dan birini gösteriyoruz.

Tanım 6.1 *A bir reel cebir ve V, F cismi üzerinde bir vektör uzayı olsun.*

$$\rho : A \to End_F(V)$$

reel lineer dönüşümü, $\forall a, b \in A$ ve 1, A cebrinin birimi olmak üzere,

$$\begin{aligned} \rho(ab) &= \rho(a) \circ \rho(b) \\ \rho(1) &= Id \end{aligned}$$

koşullarını sağlıyorsa ρ dönüşümüne A nın bir F-temsili denir. ($F = \mathbb{H}$, $V = \mathbb{H}^n$ olması durumunda, $\mathbb{H}$ değişme özelliğine sahip olmadığından $\mathbb{H}^n$, $\mathbb{H}$ üzerinde bir modüldür.)

$\rho_1 : A \longrightarrow End_F(V_1)$ ve $\rho_2 : A \longrightarrow End_F(V_2)$, A cebrinin iki temsili olsun. Her $a \in A$ için, $\rho_2(a) = f \circ \rho_1(a) \circ f^{-1}$ olacak şekilde $f : V_1 \to V_2$ lineer izomorfizması varsa ρ_1 ve ρ_2 ye denk temsiller denir.

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ \rho_1(a) \downarrow & & \downarrow \rho_2(a) \\ V_1 & \xrightarrow{f} & V_2 \end{array}$$

ρ dönüşümünün çekirdeği A cebrinde iki taraflı idealdir. $\rho(1) = Id$ olduğundan $1 \notin Ker\rho$ dur.

Yardımcı Teorem 6.1 *$F(n)$ matris cebirlerindeki her iki taraflı ideal, ya $\{0\}$ ya da tüm matris cebiridir. (Bakınız [8])*

Sonuç 6.1 *$F(n)$ nin her temsili birebirdir.*

Kanıt. $\forall a, b \in F(n)$ için $\rho(a) = \rho(b)$ olsun. $\forall v \in F^n$ için,

$$\begin{aligned}
\rho(a)(v) &= \rho(b)(v) \\
\rho(a)(v) - \rho(b)(v) &= 0 \\
\rho(a-b)(v) &= 0 \\
a - b &\in Ker(\rho)
\end{aligned}$$

elde edilir. ρ dönüşümünün çekirdeği, $F(n)$ de iki taraflı idealdir. Yardımcı teorem (6.1) den $F(n)$ matris cebirindeki iki taraflı ideal ya $\{0\}$ ya da tüm matris cebiridir. Bu durumda $a - b$ ya $\{0\}$ ya da tüm matris cebiridir. Ancak $a - b$ tüm matris cebiri olamaz. Çünkü $1 \notin Ker(\rho)$ dur. O halde $a - b = 0$ dır. Buradan da $a = b$ elde edilir. ∎

Tanım 6.2 *$a \in F(n)$ ve $x \in F^n$ için,*

$$\begin{aligned}
&\rho : F(n) \longrightarrow End_F(F^n) \\
&\rho(a)x = ax
\end{aligned}$$

şeklinde tanımlanan $F(n)$ nin bir F-temsiline, $F(n)$ nin standart temsili denir.

$a \in \mathbb{C}(n)$ verildiğinde a nın konjuge matrisini $\bar{a}$ olarak alalım. $a \in \mathbb{C}(n)$ ve $z \in \mathbb{C}^n$ olmak üzere,

$$\begin{aligned}
&\bar{\rho} : \mathbb{C}(n) \to End_{\mathbb{C}}(\mathbb{C}^n) \\
&\bar{\rho}(a)z = \bar{a}z
\end{aligned}$$

şeklinde tanımlı $\mathbb{C}(n)$ reel cebrinin $\mathbb{C}$-temsili, $\mathbb{C}(n)$ nin konjuge temsili olarak adlandırılır.

A nın her $\mathbb{C}$-temsili veya $\mathbb{H}$-temsili aynı zamanda bir $\mathbb{R}$-temsildir. $\mathbb{R}$-temsil olarak $\mathbb{C}(n)$ nin standart temsili ve konjuge temsili denktir. Ancak $\mathbb{C}(n)$ nin standart temsili ve konjuge temsili, $\mathbb{C}$-temsil olarak denk değillerdir.

Bir $\rho : A \longrightarrow End_F(V)$ temsili verildiğinde, V uzayının her $a \in A$ için $\rho(a)(W) \subset W$ olacak şekilde $\{0\}$ ve V den farklı bir W alt uzayı varsa bu temsile indirgenebilir temsil denir. Eğer bir temsil indirgenebilir değilse o temsile indirgenemez temsil denir.

Tanım 6.3 *$\rho_1 : A \longrightarrow End_F(V_1)$ ve $\rho_2 : A \longrightarrow End_F(V_2)$ A cebrinin iki temsili olsun.*

$$\rho = (\rho_1 \oplus \rho_2) : A \longrightarrow End_F(V_1 \oplus V_2)$$
$$\rho(a) = \rho_1(a) \oplus \rho_2(a)$$

şeklinde tanımlanan ρ dönüşümü A nın bir F-temsilidir. $\rho = \rho_1 \oplus \rho_2$ formunda yazılabilen ρ dönüşümünün F-temsili, F-indirgenebilirdir. Eğer A nın F-temsili indirgenebilir değilse bu temsile indirgenemez F-temsil denir.

Teorem 6.1 *F^n üzerindeki $F(n)$ nin temsili, $F(n)$ nin indirgenemez $\mathbb{R}$-temsilidir.*

Sonuç 6.2 *$\mathbb{C}(n)$ reel cebrinin standart $\mathbb{C}$-temsili ve konjuge $\mathbb{C}$-temsili olmak üzere iki indirgenemez $\mathbb{C}$-temsili vardır.*

Sonuç 6.3 *$\mathbb{H}(n)$ nin $\mathbb{H}^n$ üzerindeki standart temsili, indirgenemez $\mathbb{H}$-temsildir.*

Yardımcı Teorem 6.2 *$F = \mathbb{R}$ veya $\mathbb{H}$ olmak üzere $F(n) \oplus F(n)$ cebrinin iki indirgenemez F-temsili vardır. Bunlar $(a,b) \in F(n) \oplus F(n)$ ve her $x \in F^n$ olmak üzere,*

$$\rho_1(a,b)(x) = ax$$
$$\rho_2(a,b)(x) = bx$$

şeklindedir.

Bu durumda Bölüm 5 de elde edilen tabloyu

$p-q \pmod 8$	$p+q$	$Cl_{p,q}$
$0,2$	$2k$	$\mathbb{R}(2^k)$
1	$2k+1$	$\mathbb{R}(2^k) \oplus \mathbb{R}(2^k)$
$3,7$	$2k+1$	$\mathbb{C}(2^k)$
$4,6$	$2k+2$	$\mathbb{H}(2^k)$
5	$2k+3$	$\mathbb{H}(2^k) \oplus \mathbb{H}(2^k)$

şeklinde ifade edelim. Bu son tabloya bakarak aşağıdaki sonuçlar elde edilir.

1) $p-q = 0,2 \pmod 8$ durumunda $Cl_{p,q} \cong \mathbb{R}(2^k)$ olduğundan Bölüm 5 de verilen izomorfizmler $Cl_{p,q}$ için indirgenemez $\mathbb{R}$−temsillerdir.

2) $p-q=1 \pmod 8$ durumunda $Cl_{p,q} \cong \mathbb{R}\left(2^k\right) \oplus \mathbb{R}\left(2^k\right)$ olduğundan Bölüm 5 de verilen izomorfizm ve sonrada $\mathbb{R}\left(2^k\right) \oplus \mathbb{R}\left(2^k\right)$ den $\mathbb{R}\left(2^k\right)$ ye giden birinci izdüşüm kullanılırsa bu tip Clifford cebirleri için indirgenemez $\mathbb{R}-$temsiller elde edilmiş olur. Eğer $\mathbb{R}\left(2^k\right) \oplus \mathbb{R}\left(2^k\right)$ den $\mathbb{R}\left(2^k\right)$ ya ikinci izdüşüm kullanılırsa öncekine denk olmayan indirgenemez $\mathbb{R}-$temsiller elde edilmiş olur.

3) $p-q=3,7 \pmod 8$ durumunda $Cl_{p,q} \cong \mathbb{C}\left(2^k\right)$ olduğundan Bölüm 5 de verilen izomorfizmler bu tip Clifford cebirleri için indirgenemez $\mathbb{C}-$temsillerdir.

4) $p-q=4,6 \pmod 8$ durumunda $Cl_{p,q} \cong \mathbb{H}\left(2^k\right)$ olduğundan Bölüm 5 de verilen izomorfizmler bu tip Clifford cebirleri için indirgenemez $\mathbb{H}-$temsillerdir.

5) $p-q=5 \pmod 8$ durumunda $Cl_{p,q} \cong \mathbb{H}\left(2^k\right) \oplus \mathbb{H}\left(2^k\right)$ olduğundan Bölüm 5 de verilen izomorfizm ve sonrada $\mathbb{H}\left(2^k\right) \oplus \mathbb{H}\left(2^k\right)$ den $\mathbb{H}\left(2^k\right)$ ye giden birinci izdüşüm kullanılırsa bu tip Clifford cebirleri için indirgenemez $\mathbb{H}-$temsiller elde edilmiş olur. Eğer $\mathbb{H}\left(2^k\right) \oplus \mathbb{H}\left(2^k\right)$ den $\mathbb{H}\left(2^k\right)$ ya ikinci izdüşüm kullanılırsa öncekine denk olmayan indirgenemez $\mathbb{H}-$temsiller elde edilmiş olur.

7 $Cl_{p,q}$ CLİFFORD CEBRİNİN REEL TEMSİLLERİNİN ELDE EDİLMESİ

Önceki bölümde $Cl_{p,q}$ Clifford cebirlerinin indirgenemez $F-$temsillerini verdik. Bu bölümdeki amacımız $Cl_{p,q}$ Clifford cebirlerinin indirgenemez reel temsillerini elde etmektir. [4] de $Cl_{0,n}$ tipindeki Clifford cebirlerinin indirgenemez temsilleri elde edilmiştir. Bu temsilleri ve önceki bölümdeki yardımcı teorem (6.2) yi kullanarak diğer $Cl_{p,q}$ tipindeki Clifford cebirlerinin indirgenemez temsillerini bulacağız.

A_1, A_2 iki reel cebir ve $\rho_1 : A_1 \to End_{\mathbb{R}}(V_1)$, $\rho_2 : A_2 \to End_{\mathbb{R}}(V_2)$ bu cebirlerin temsilleri olsunlar, bu şekildeki dönüşümler yardımıyla bölüm 2 de $\rho_1 \otimes \rho_2 : A_1 \otimes A_2 \to End_{\mathbb{R}}(V_1 \otimes V_2)$ şeklinde tek türlü bir dönüşüm tanımlandığını söylemiştik. Bu dönüşüme ρ_1 ve ρ_2 temsillerinin tensör çarpımı denir. $\rho_1 \otimes \rho_2$ dönüşümü de $A_1 \otimes A_2$ cebrinin $V_1 \otimes V_2$ vektör uzayı üzerinde bir temsilidir. $\rho_1 : A_1 \to End_{\mathbb{R}}(V_1)$, $\rho_2 : A_2 \to End_{\mathbb{R}}(V_2)$ indirgenemez temsiller iken $\rho_1 \otimes \rho_2$ temsili indirgenemez bir temsil olmak zorunda değildir. Örneğin ρ_1 ve ρ_2 yi $\mathbb{H}$ nın $\mathbb{R}^4$ üzerindeki standart reel temsilini alırsak, bunların tensör çarpımı

$$\rho_1 \otimes \rho_2 : \mathbb{H} \otimes \mathbb{H} \to End_{\mathbb{R}}(\mathbb{R}^4 \otimes \mathbb{R}^4) \cong End_{\mathbb{R}}(\mathbb{R}^{16})$$

şeklinde bir cebir homomorfizmi verir. Ancak $\mathbb{H} \otimes \mathbb{H} \cong \mathbb{R}(4)$ olduğundan, $\mathbb{H} \otimes \mathbb{H}$ nin indirgenemez modülü $\mathbb{R}^4$ olmak zorundadır. Dolayısıyla $\rho_1 \otimes \rho_2$ indirgenemez bir temsil değildir. Bazı durumlarda indirgenemez iki temsilin tensör çarpımı da indirgenemez bir temsil olur.

Yardımcı Teorem 7.1 *i)* $\rho_1 : F(n) \to End_{\mathbb{R}}(F^n)$ *ve* $\rho_2 : \mathbb{R}(m) \to End_{\mathbb{R}}(\mathbb{R}^m)$ *indirgenemez temsillerinin tensör çarpımı indirgenemez bir temsildir.*

ii) $\rho_1 : F(n) \oplus F(n) \to End_{\mathbb{R}}(F^n)$ *ve* $\rho_2 : \mathbb{R}(m) \to End_{\mathbb{R}}(\mathbb{R}^m)$ *indirgenemez temsillerinin tensör çarpımı indirgenemez bir temsildir.*

Bu temsillerin tensör çarpımı $\rho_1 \otimes \rho_2 : F(n) \otimes \mathbb{R}(m) \to End_{\mathbb{R}}(F^n \otimes \mathbb{R}^m)$ şeklinde bir homomorfizmdir.

$$F(n) \otimes \mathbb{R}(m) \cong F(nm) \text{ ve } End_{\mathbb{R}}(F^n \otimes \mathbb{R}^m) \cong End_{\mathbb{R}}(F^{nm})$$

izomorfizmlerinden ve $F(nm)$ nin F^{nm} üzerindeki reel temsili indirgenemez olduğundan $\rho_1 \otimes \rho_2$ indirgenemez bir temsildir.

Öncelikle $Cl_{n+2,0} \cong Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{0,n} \otimes \mathbb{R}(2)$ izomorf olma ilişkisinden yararlanarak $Cl_{m,0}$ tipindeki Clifford cebirlerinin indirgenemez temsillerini bulabiliriz. $\rho_{0,n}$ dönüşümü $Cl_{0,n}$ nin [4] de verilen indirgenemez temsili, ρ da $\mathbb{R}(2)$ nin $\mathbb{R}^2$ üzerindeki standart temsili olsun. Bu durumda $\rho_{0,n} \otimes \rho$ tensör çarpımı $Cl_{n+2,0}$ nin indirgenemez bir temsilini verir.

Benzer şekilde $Cl_{p+1,q+1} \cong Cl_{p,q} \otimes Cl_{1,1} \cong Cl_{p,q} \otimes \mathbb{R}(2)$ izomorf olma ilişkisinden yararlanarak tüm $Cl_{r,s}$ tipindeki Clifford cebirlerinin indirgenemez temsillerini bulabiliriz. Başlangıçta $p = 0$ ise $Cl_{p,q}$ nun indirgenemez temsili $\rho_{p,q}$ [4] de verilmiş, $q = 0$ ise $Cl_{p,q}$ nun indirgenemez temsili $\rho_{p,q}$ nun nasıl elde edileceğini yukarıda söyledik. $\rho_{p,q}$ ile $\mathbb{R}(2)$ nin indirgenemez temsilinin tensör çarpımı $Cl_{p+1,q+1}$ cebrinin indirgenemez bir temsilini verir. Bu şekilde tüm $Cl_{r,s}$ Clifford cebirlerinin indirgenemez temsilleri elde edilmiş olur.

Şimdi yukarıda belirtilen yöntemle tablodaki Clifford cebirlerinin indirgenemez temsillerini listeliyoruz (Aslında söz konusu $\mathbb{R}$-lineer endomorfizmlerin ilgili uzaydaki standart tabana göre matrisini veriyoruz.).

$n = 1$ **için,**

$$
\begin{array}{lcl}
Cl_{1,0} & \rightarrow & \mathbb{R} \\
e_1 & \mapsto & -1 \\
Cl_{0,1} & \rightarrow & \mathbb{R}(2) \\
\varepsilon_1 & \mapsto & \sigma_1\sigma_2
\end{array}
$$

$n = 2$ **için,**

$$
\begin{array}{lcl}
Cl_{2,0} & \rightarrow & \mathbb{R}(2) \\
e_1 & \mapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \\
e_2 & \mapsto & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_2 \\
Cl_{1,1} & \rightarrow & \mathbb{R}(2) \\
e_1 & \mapsto & \sigma_1 \\
\varepsilon_1 & \mapsto & \sigma_1\sigma_2
\end{array}
$$

$$
\begin{array}{lll}
Cl_{0,2} & \to & \mathbb{R}(4) \\
\varepsilon_1 & \mapsto & I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \sigma_1\sigma_2\otimes\sigma_2
\end{array}
$$

$n = 3$ **için,**

$$
\begin{array}{lllll}
Cl_{3,0} & \to & Cl_{0,1}\otimes Cl_{2,0} & \to & \mathbb{R}(4) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes\sigma_1 \\
e_2 & \mapsto & 1\otimes e_2 & \mapsto & I\otimes\sigma_2 \\
e_3 & \mapsto & \varepsilon_1\otimes e_1e_2 & \mapsto & \sigma_1\sigma_2\otimes\sigma_1\sigma_2 \\
Cl_{2,1} & \to & Cl_{1,0}\otimes Cl_{1,1} & \to & \mathbb{R}\otimes\mathbb{R}(2)\cong\mathbb{R}(2) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & 1\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}=\sigma_1 \\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & -1\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}-1&0\\0&1\end{bmatrix}=-\sigma_2 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & 1\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}=\sigma_1\sigma_2 \\
Cl_{1,2} & \to & Cl_{0,1}\otimes Cl_{1,1} & \to & \mathbb{R}(2)\otimes\mathbb{R}(2)\cong\mathbb{R}(4) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix}=I\otimes\sigma_1 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & \begin{bmatrix}1&0\\0&1\end{bmatrix}\otimes\begin{bmatrix}0&-1\\1&0\end{bmatrix}=I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & \begin{bmatrix}0&-1\\1&0\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\sigma_1\sigma_2\otimes\sigma_2 \\
Cl_{0,3} & \to & \mathbb{R}(4) & & \\
\varepsilon_1 & \mapsto & I\otimes\sigma_1\sigma_2 & & \\
\varepsilon_2 & \mapsto & \sigma_1\sigma_2\otimes\sigma_2 & & \\
\varepsilon_3 & \mapsto & \sigma_1\sigma_2\otimes\sigma_1 & &
\end{array}
$$

$n = 4$ için,

$$
\begin{aligned}
Cl_{4,0} &\rightarrow Cl_{0,2} \otimes Cl_{2,0} \rightarrow \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
e_1 &\mapsto I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes \sigma_2 \\
e_3 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_4 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{3,1} &\rightarrow Cl_{2,0} \otimes Cl_{1,1} \rightarrow \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \\
e_1 &\mapsto 1 \otimes e_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I \otimes \sigma_1 \\
e_2 &\mapsto e_1 \otimes e_1\varepsilon_1 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto e_2 \otimes e_1\varepsilon_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto 1 \otimes \varepsilon_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \sigma_2 \otimes \sigma_1\sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{2,2} &\rightarrow Cl_{1,1} \otimes Cl_{1,1} \rightarrow \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \\
e_1 &\mapsto I \otimes \sigma_1 \\
e_2 &\mapsto \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{1,3} &\rightarrow Cl_{0,2} \otimes Cl_{1,1} \rightarrow \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
e_1 &\mapsto 1 \otimes e_1 \mapsto I \otimes I \otimes \sigma_1 \\
\varepsilon_1 &\mapsto 1 \otimes \varepsilon_1 \mapsto I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto \varepsilon_1 \otimes e_1\varepsilon_1 \mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto \varepsilon_2 \otimes e_1\varepsilon_1 \mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{0,4} &\rightarrow \mathbb{R}(8) \\
\varepsilon_1 &\mapsto I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \\
\varepsilon_4 &\mapsto \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_1
\end{aligned}
$$

$n = 5$ için,

$$Cl_{5,0} \rightarrow Cl_{0,3} \otimes Cl_{2,0} \rightarrow \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8)$$

$$\begin{aligned}
e_1 &\mapsto I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes \sigma_2 \\
e_3 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_4 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \\
e_5 &\mapsto \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2
\end{aligned}$$

$$Cl_{4,1} \rightarrow Cl_{3,0} \otimes Cl_{1,1} \rightarrow \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8)$$

$$\begin{aligned}
e_1 &\mapsto 1 \otimes e_1 &&\mapsto I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto e_1 \otimes e_1\varepsilon_1 &&\mapsto I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto e_2 \otimes e_1\varepsilon_1 &&\mapsto I \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto e_3 \otimes e_1\varepsilon_1 &&\mapsto \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto \varepsilon_1 \otimes e_1\varepsilon_1 &&\mapsto I \otimes I \otimes \sigma_1\sigma_2
\end{aligned}$$

$$Cl_{3,2} \rightarrow Cl_{2,1} \otimes Cl_{1,1} \rightarrow \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4)$$

$$\begin{aligned}
e_1 &\mapsto 1 \otimes e_1 &&\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = I \otimes \sigma_1 \\
e_2 &\mapsto e_1 \otimes e_1\varepsilon_1 &&\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto e_2 \otimes e_1\varepsilon_1 &&\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto 1 \otimes \varepsilon_1 &&\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto \varepsilon_1 \otimes e_1\varepsilon_1 &&\mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_1\sigma_2 \otimes \sigma_2
\end{aligned}$$

$$Cl_{2,3} \rightarrow Cl_{1,2} \otimes Cl_{1,1} \rightarrow \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8)$$

$$\begin{aligned}
e_1 &\mapsto 1 \otimes e_1 &&\mapsto I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto e_1 \otimes e_1\varepsilon_1 &&\mapsto I \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto 1 \otimes \varepsilon_1 &&\mapsto I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto \varepsilon_1 \otimes e_1\varepsilon_1 &&\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto \varepsilon_2 \otimes e_1\varepsilon_1 &&\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}$$

$$
\begin{array}{lllll}
Cl_{1,4} & \rightarrow & Cl_{0,3} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes \sigma_1 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2
\end{array}
$$

$$
\begin{array}{lll}
Cl_{0,5} & \rightarrow & \mathbb{R}(8) \\
\varepsilon_1 & \mapsto & -\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes I \\
\varepsilon_3 & \mapsto & -\sigma_1 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \\
\varepsilon_4 & \mapsto & -\sigma_1 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \\
\varepsilon_5 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes I
\end{array}
$$

$n = 6$ **için,**

$$
\begin{array}{lllll}
Cl_{6,0} & \rightarrow & Cl_{0,4} \otimes Cl_{2,0} & \rightarrow & \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_4 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \\
e_5 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \\
e_6 & \mapsto & \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{5,1} & \rightarrow & Cl_{4,0} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{4,2} & \rightarrow & Cl_{3,1} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
e_1 & \mapsto & I \otimes I \otimes \sigma_1 & & \\
e_2 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 & & \\
e_3 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 & & \\
e_4 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 & & \\
\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 & & \\
\varepsilon_2 & \mapsto & \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 & & \\
Cl_{3,3} & \rightarrow & Cl_{2,2} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(4) \otimes \mathbb{R}(2) \cong \mathbb{R}(8) \\
e_1 & \mapsto & I \otimes I \otimes \sigma_1 & & \\
e_2 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 & & \\
e_3 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 & & \\
\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 & & \\
\varepsilon_2 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 & & \\
\varepsilon_3 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 & & \\
Cl_{2,4} & \rightarrow & Cl_{1,3} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
Cl_{1,5} & \rightarrow & Cl_{0,4} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2
\end{array}
$$

$$
\begin{aligned}
Cl_{0,6} &\rightarrow \mathbb{R}(16) \\
\varepsilon_1 &\mapsto -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I \\
\varepsilon_2 &\mapsto -\sigma_1\sigma_2 \otimes I \otimes I \\
\varepsilon_3 &\mapsto -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \\
\varepsilon_4 &\mapsto -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_5 &\mapsto -\sigma_1 \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_6 &\mapsto -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2
\end{aligned}
$$

$n = 7$ için,

$$
\begin{aligned}
Cl_{7,0} &\rightarrow Cl_{0,5} \otimes Cl_{2,0} \rightarrow \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes \sigma_2 \\
e_3 &\mapsto -\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_4 &\mapsto -\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
e_5 &\mapsto -\sigma_1 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_6 &\mapsto -\sigma_1 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_7 &\mapsto \sigma_1 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_1\sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{6,1} &\rightarrow Cl_{5,0} \otimes Cl_{1,1} \rightarrow \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{5,2} &\rightarrow Cl_{4,1}\otimes Cl_{1,1} \rightarrow \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16)\\
e_1 &\mapsto I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto I\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_5 &\mapsto \sigma_1\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
Cl_{4,3} &\rightarrow Cl_{3,2}\otimes Cl_{1,1} \rightarrow \mathbb{R}(4)\otimes\mathbb{R}(2)\cong\mathbb{R}(8)\\
e_1 &\mapsto I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto \sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto -\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
Cl_{3,4} &\rightarrow Cl_{2,3}\otimes Cl_{1,1} \rightarrow \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16)\\
e_1 &\mapsto 1\otimes e_1 \mapsto I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto e_1\otimes e_1\varepsilon_1 \mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto e_2\otimes e_1\varepsilon_1 \mapsto I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto 1\otimes\varepsilon_1 \mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto \varepsilon_1\otimes e_1\varepsilon_1 \mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto \varepsilon_2\otimes e_1\varepsilon_1 \mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 &\mapsto \varepsilon_3\otimes e_1\varepsilon_1 \mapsto \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{aligned}
$$

$$
\begin{array}{lllll}
Cl_{2,5} & \to & Cl_{1,4}\otimes Cl_{1,1} & \to & \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1 \\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes\sigma_1\otimes\sigma_2 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2 \\
Cl_{1,6} & \to & Cl_{0,5}\otimes Cl_{1,1} & \to & \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16)
\end{array}
$$

$$
\begin{array}{lllll}
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2\otimes I\otimes I\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\otimes\sigma_1\sigma_2\otimes I\otimes\sigma_2
\end{array}
$$

$$
\begin{array}{lll}
Cl_{0,7} & \to & \mathbb{R}(8) \\
\varepsilon_1 & \mapsto & -\sigma_2\otimes\sigma_1\sigma_2\otimes I \\
\varepsilon_2 & \mapsto & -\sigma_1\sigma_2\otimes I\otimes I \\
\varepsilon_3 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1 \\
\varepsilon_4 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \sigma_1\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_6 & \mapsto & -\sigma_2\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_7 & \mapsto & \sigma_2\otimes\sigma_2\otimes\sigma_1\sigma_2
\end{array}
$$

$n = 8$ **için,**

$$
\begin{aligned}
Cl_{8,0} &\rightarrow Cl_{0,6} \otimes Cl_{2,0} \rightarrow \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes \sigma_2 \\
e_3 &\mapsto -\sigma_2 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \\
e_4 &\mapsto -\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
e_5 &\mapsto -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \\
e_6 &\mapsto -\sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \\
e_7 &\mapsto -\sigma_1 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
e_8 &\mapsto -\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \\
Cl_{7,1} &\rightarrow Cl_{6,0} \otimes Cl_{1,1} \rightarrow \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_7 &\mapsto \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
Cl_{6,2} &\rightarrow Cl_{5,1} \otimes Cl_{1,1} \rightarrow \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{5,3} &\rightarrow Cl_{4,2}\otimes Cl_{1,1} \rightarrow \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16)\\
e_1 &\mapsto I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto \sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_5 &\mapsto \sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto \sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
Cl_{4,4} &\rightarrow Cl_{3,3}\otimes Cl_{1,1} \rightarrow \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16)\\
e_1 &\mapsto I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto \sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 &\mapsto \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
Cl_{3,5} &\rightarrow Cl_{2,4}\otimes Cl_{1,1} \rightarrow \mathbb{R}(16)\otimes\mathbb{R}(2)\cong\mathbb{R}(32)\\
e_1 &\mapsto 1\otimes e_1 \mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto e_1\otimes e_1\varepsilon_1 \mapsto I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto e_2\otimes e_1\varepsilon_1 \mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto 1\otimes\varepsilon_1 \mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto \varepsilon_1\otimes e_1\varepsilon_1 \mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto \varepsilon_2\otimes e_1\varepsilon_1 \mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 &\mapsto \varepsilon_3\otimes e_1\varepsilon_1 \mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_5 &\mapsto \varepsilon_4\otimes e_1\varepsilon_1 \mapsto \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
Cl_{2,6} &\rightarrow Cl_{1,5}\otimes Cl_{1,1} \rightarrow \mathbb{R}(16)\otimes\mathbb{R}(2)\cong\mathbb{R}(32)\\
e_1 &\mapsto 1\otimes e_1 \mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto e_1\otimes e_1\varepsilon_1 \mapsto I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2
\end{aligned}
$$

$$
\begin{array}{lllll}
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & \sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{1,7} & \to & Cl_{0,6}\otimes Cl_{1,1} & \to & \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes\sigma_1\sigma_2\otimes I\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2\otimes I\otimes I\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2
\end{array}
$$

$$
\begin{array}{lll}
Cl_{0,8} & \to & \mathbb{R}(16) \\
\varepsilon_1 & \mapsto & -I\otimes I\otimes\sigma_2\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & -I\otimes I\otimes\sigma_1\sigma_2\otimes I \\
\varepsilon_3 & \mapsto & -I\otimes\sigma_1\otimes\sigma_1\otimes\sigma_1\sigma_2 \\
\varepsilon_4 & \mapsto & -I\otimes\sigma_2\otimes\sigma_1\otimes\sigma_1\sigma_2 \\
\varepsilon_5 & \mapsto & I\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes I \\
\varepsilon_6 & \mapsto & -I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_1 \\
\varepsilon_7 & \mapsto & \sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_8 & \mapsto & \sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$n = 9$ **için,**

$$
\begin{array}{lll}
Cl_{8,1} & \rightarrow & Cl_{7,0} \otimes Cl_{1,1} \rightarrow \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes I \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & -\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & -\sigma_1 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & -\sigma_1 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
Cl_{7,2} & \rightarrow & Cl_{6,1} \otimes Cl_{1,1} \rightarrow \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & I \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
Cl_{6,3} & \rightarrow & Cl_{5,2} \otimes Cl_{1,1} \rightarrow \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$\begin{array}{lll}
Cl_{5,4} & \rightarrow & Cl_{4,3} \otimes Cl_{1,1} \rightarrow \mathbb{R}(8) \otimes \mathbb{R}(2) \cong \mathbb{R}(16) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

$$\begin{array}{lllll}
Cl_{4,5} & \rightarrow & Cl_{3,4} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

$$\begin{array}{lllll}
Cl_{3,6} & \rightarrow & Cl_{2,5} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}$$

$$
\begin{array}{lllll}
Cl_{2,7} & \rightarrow & Cl_{1,6}\otimes Cl_{1,1} & \rightarrow & \mathbb{R}(16)\otimes\mathbb{R}(2)\cong\mathbb{R}(32) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1 \\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2\otimes I\otimes I\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\otimes\sigma_1\sigma_2\otimes I\otimes\sigma_2\otimes\sigma_2 \\
Cl_{1,8} & \rightarrow & Cl_{0,7}\otimes Cl_{1,1} & \rightarrow & \mathbb{R}(8)\otimes\mathbb{R}(2)\cong\mathbb{R}(16) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes\sigma_1\sigma_2\otimes I\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2\otimes I\otimes I\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7\otimes e_1\varepsilon_1 & \mapsto & \sigma_2\otimes\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2
\end{array}
$$

$n = 10$ **için,**

$$
\begin{array}{lllll}
Cl_{8,2} & \rightarrow & Cl_{7,1}\otimes Cl_{1,1} & \rightarrow & \mathbb{R}(32)\otimes\mathbb{R}(2)\cong\mathbb{R}(64) \\
e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1 \\
e_2 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2 \\
e_3 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2 \\
e_4 & \mapsto & I\otimes I\otimes I\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
e_5 & \mapsto & I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$$
\begin{aligned}
e_6 &\mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_7 &\mapsto \sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_8 &\mapsto \sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2
\end{aligned}
$$

$$
Cl_{7,3} \;\rightarrow\; Cl_{6,2}\otimes Cl_{1,1} \;\rightarrow\; \mathbb{R}(32)\otimes\mathbb{R}(2)\cong\mathbb{R}(64)
$$

$$
\begin{aligned}
e_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_5 &\mapsto I\otimes I\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_6 &\mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_7 &\mapsto \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{aligned}
$$

$$
Cl_{6,4} \;\rightarrow\; Cl_{5,3}\otimes Cl_{1,1} \;\rightarrow\; \mathbb{R}(16)\otimes\mathbb{R}(2)\cong\mathbb{R}(32)
$$

$$
\begin{aligned}
e_1 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_5 &\mapsto \sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_6 &\mapsto \sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 &\mapsto \sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{aligned}
$$

$$
\begin{array}{lll}
Cl_{5,5} & \rightarrow & Cl_{4,4}\otimes Cl_{1,1} \;\rightarrow\; \mathbb{R}(16)\otimes\mathbb{R}(2)\cong\mathbb{R}(32)\\
e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 & \mapsto & I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 & \mapsto & I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_5 & \mapsto & \sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 & \mapsto & I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 & \mapsto & I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_5 & \mapsto & \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{4,6} & \rightarrow & Cl_{3,5}\otimes Cl_{1,1} & \rightarrow & \mathbb{R}(32)\otimes\mathbb{R}(2)\cong\mathbb{R}(64)\\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 & \mapsto & e_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 & \mapsto & e_3\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$$
\begin{aligned}
Cl_{3,7} &\rightarrow Cl_{2,6}\otimes Cl_{1,1} \rightarrow \mathbb{R}(32)\otimes\mathbb{R}(2)\cong\mathbb{R}(64)\\
e_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_5 &\mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_6 &\mapsto \sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_7 &\mapsto \sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{aligned}
$$

$$
\begin{array}{lclcl}
Cl_{2,8} & \longrightarrow & Cl_{1,7}\otimes Cl_{1,1} & \longrightarrow & \mathbb{R}(16)\otimes\mathbb{R}(2)\cong\mathbb{R}(32)\\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes\sigma_1\sigma_2\otimes I\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2\otimes I\otimes I\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_7 & \mapsto & \varepsilon_6\otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_8 & \mapsto & \varepsilon_7\otimes e_1\varepsilon_1 & \mapsto & -\sigma_2\otimes\sigma_1\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$n = 11$ için,

$$\begin{aligned}
Cl_{8,3} &\rightarrow Cl_{7,2} \otimes Cl_{1,1} \rightarrow \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto I \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 &\mapsto \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
Cl_{7,4} &\rightarrow Cl_{6,3} \otimes Cl_{1,1} \rightarrow \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 &\mapsto \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 &\mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}$$

$$
\begin{aligned}
Cl_{6,5} &\rightarrow Cl_{5,4} \otimes Cl_{1,1} \rightarrow \mathbb{R}(16) \otimes \mathbb{R}(2) \cong \mathbb{R}(32) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{5,6} &\rightarrow Cl_{4,5} \otimes Cl_{1,1} \rightarrow \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 &\mapsto 1 \otimes e_1 \mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto e_1 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto e_2 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto e_3 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto e_4 \otimes e_1\varepsilon_1 \mapsto I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto 1 \otimes \varepsilon_1 \mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto \varepsilon_1 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto \varepsilon_2 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 &\mapsto \varepsilon_3 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 &\mapsto \varepsilon_4 \otimes e_1\varepsilon_1 \mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 &\mapsto \varepsilon_5 \otimes e_1\varepsilon_1 \mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

$$
\begin{aligned}
Cl_{4,7} &\rightarrow Cl_{3,6} \otimes Cl_{1,1} \rightarrow \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 &\mapsto 1 \otimes e_1 \mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto e_1 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto e_2 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto e_3 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

$$
\begin{array}{lllll}
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{3,8} & \rightarrow & Cl_{2,7} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_2 \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1\sigma_2 \otimes I \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1 \otimes \sigma_1 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & -\sigma_1 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1 \otimes \sigma_1\sigma_2 \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$n = 12$ için,

$$
\begin{array}{lllll}
Cl_{8,4} & \rightarrow & Cl_{7,3} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128)
\end{array}
$$

$$
\begin{array}{lll}
e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & I \otimes I \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$
\begin{aligned}
e_8 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
Cl_{7,5} &\rightarrow Cl_{6,4} \otimes Cl_{1,1} \rightarrow \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 &\mapsto \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 &\mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 &\mapsto \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
Cl_{6,6} &\rightarrow Cl_{5,5} \otimes Cl_{1,1} \rightarrow \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 &\mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

$$
\begin{array}{lllll}
Cl_{5,7} & \rightarrow & Cl_{4,6}\otimes Cl_{1,1} & \rightarrow & \mathbb{R}(64)\otimes\mathbb{R}(2)\cong\mathbb{R}(128) \\
e_1 & \mapsto & 1\otimes e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1 \\
e_2 & \mapsto & e_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2 \\
e_3 & \mapsto & e_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2 \\
e_4 & \mapsto & e_3\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
e_5 & \mapsto & e_4\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_1 & \mapsto & 1\otimes\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4\otimes e_1\varepsilon_1 & \mapsto & I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5\otimes e_1\varepsilon_1 & \mapsto & I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6\otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$n = 13$ **için,**

$$
\begin{array}{lllll}
Cl_{8,5} & \rightarrow & Cl_{7,4}\otimes Cl_{1,1} & \rightarrow & \mathbb{R}(64)\otimes\mathbb{R}(2)\cong\mathbb{R}(128)
\end{array}
$$

$$
\begin{array}{lll}
e_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1 \\
e_2 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2 \\
e_3 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2 \\
e_4 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
e_5 & \mapsto & I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
e_6 & \mapsto & I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
e_7 & \mapsto & I\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
e_8 & \mapsto & \sigma_1\sigma_2\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_1 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2 \\
\varepsilon_3 & \mapsto & I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_4 & \mapsto & I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2 \\
\varepsilon_5 & \mapsto & I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{array}
$$

$$
\begin{array}{lll}
Cl_{7,6} & \rightarrow & Cl_{6,5} \otimes Cl_{1,1} \rightarrow \mathbb{R}(32) \otimes \mathbb{R}(2) \cong \mathbb{R}(64) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{6,7} & \rightarrow & Cl_{5,6} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{5,8} & \rightarrow & Cl_{4,7} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \varepsilon_5 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \varepsilon_6 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 & \mapsto & \varepsilon_7 \otimes e_1\varepsilon_1 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$n = 14$ **için,**

$$
\begin{array}{lll}
Cl_{8,6} & \rightarrow & Cl_{7,5} \otimes Cl_{1,1} \rightarrow \mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & \sigma_2 \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$
\begin{aligned}
Cl_{7,7} &\rightarrow Cl_{6,6}\otimes Cl_{1,1} \rightarrow \mathbb{R}(64)\otimes\mathbb{R}(2)\cong\mathbb{R}(128)\\
e_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_3 &\mapsto I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_5 &\mapsto I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_6 &\mapsto I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_7 &\mapsto \sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_5 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_6 &\mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_7 &\mapsto \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
Cl_{6,8} &\rightarrow Cl_{5,7}\otimes Cl_{1,1} \rightarrow \mathbb{R}(128)\otimes\mathbb{R}(2)\cong\mathbb{R}(256)\\
e_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\\
e_2 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\\
e_3 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\\
e_4 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_5 &\mapsto I\otimes I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
e_6 &\mapsto I\otimes I\otimes\sigma_1\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_1 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\\
\varepsilon_2 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\\
\varepsilon_3 &\mapsto I\otimes I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_4 &\mapsto I\otimes I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_5 &\mapsto I\otimes I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_6 &\mapsto I\otimes I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_7 &\mapsto I\otimes\sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\\
\varepsilon_8 &\mapsto \sigma_1\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2
\end{aligned}
$$

$n = 15$ **için,**

$$
\begin{array}{lll}
Cl_{8,7} & \rightarrow & Cl_{7,6} \otimes Cl_{1,1} \rightarrow \mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128) \\
e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 & \mapsto & -\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 & \mapsto & I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 & \mapsto & \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$
\begin{array}{lllll}
Cl_{7,8} & \rightarrow & Cl_{6,7} \otimes Cl_{1,1} & \rightarrow & \mathbb{R}(128) \otimes \mathbb{R}(2) \cong \mathbb{R}(256) \\
e_1 & \mapsto & 1 \otimes e_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 & \mapsto & e_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 & \mapsto & e_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 & \mapsto & e_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 & \mapsto & e_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 & \mapsto & e_5 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 & \mapsto & e_6 \otimes e_1\varepsilon_1 & \mapsto & I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 & \mapsto & 1 \otimes \varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 & \mapsto & \varepsilon_1 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 & \mapsto & \varepsilon_2 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 & \mapsto & \varepsilon_3 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 & \mapsto & \varepsilon_4 \otimes e_1\varepsilon_1 & \mapsto & I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{array}
$$

$$
\begin{aligned}
\varepsilon_6 &\mapsto \varepsilon_5 \otimes e_1\varepsilon_1 \mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 &\mapsto \varepsilon_6 \otimes e_1\varepsilon_1 \mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 &\mapsto \varepsilon_7 \otimes e_1\varepsilon_1 \mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

$n = 16$ **için,**

$$
\begin{aligned}
Cl_{8,8} &\rightarrow Cl_{7,7} \otimes Cl_{1,1} \rightarrow \mathbb{R}(64) \otimes \mathbb{R}(2) \cong \mathbb{R}(128) \\
e_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \\
e_2 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \\
e_3 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_4 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_5 &\mapsto I \otimes I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_6 &\mapsto I \otimes I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_7 &\mapsto I \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
e_8 &\mapsto \sigma_1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_1 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \\
\varepsilon_2 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \\
\varepsilon_3 &\mapsto I \otimes I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_4 &\mapsto I \otimes I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_5 &\mapsto I \otimes I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_6 &\mapsto I \otimes I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_7 &\mapsto I \otimes \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \\
\varepsilon_8 &\mapsto \sigma_1\sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
$$

8 DEJENERE CLİFFORD CEBİRLERİ

Dördüncü bölümde $\mathbb{R}^n$ üzerindeki

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \quad (n = p + q)$$

şeklinde dejenere olmayan kuadratik forma karşılık gelen $Cl_{p,q}$ Clifford cebirlerini inceledik. Bu bölümde $\mathbb{R}^n$ üzerindeki

$$Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \quad (n = p + q + r, r > 0) \qquad (8.1)$$

şeklinde dejenere kuadratik forma karşılık gelen Clifford cebirlerini inceleyeceğiz. Bu tip bir kuadratik forma karşılık gelen Clifford cebri $Cl_{p,q,r}$ şeklinde gösterilir ve buradaki r doğal sayısı kuadratik formun dejenerelik mertebesini göstermektedir

8.1 Dejenere Clifford cebirlerinin dejenere olmayan Clifford cebirleri içerisine gömülmesi

Bu tip Clifford cebirlerini de dejenere olmayan durumda olduğu gibi teorem yardımıyla belirlemeye çalışacağız.

$\mathbb{R}^n$ üzerindeki (8.1) deki dejenere kuadratik formuna karşılık gelen Sylvester tabanını

$$\{e_1, e_2, \ldots, e_p, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q, \theta_1, \ldots, \theta_r\}$$

şeklinde gösterelim. Bu durumda herhangi farklı iki taban elemanı anti-commute özelliğindedir yani x, y bu tabana ait herhangi farklı iki eleman iken $xy = -yx$ olur, ayrıca

$$\begin{aligned} 1 \leq i \leq p \quad &\text{için} \quad e_i^2 = Q(e_i) \cdot 1 = 1 \\ 1 \leq j \leq q \quad &\text{için} \quad \varepsilon_j^2 = Q(\varepsilon_j) \cdot 1 = -1 \\ 1 \leq k \leq r \quad &\text{için} \quad \theta_k^2 = Q(\theta_k) \cdot 1 = 0 \end{aligned}$$

olur. Teorem (3.5) den dolayı biliyoruz ki $\mathbb{R}^n$ üzerindeki $Q(x)$ dejenere kuadratik formuna karşılık gelen Clifford cebirleri bu tip elemanlar tarafından üretilir. Şimdi düşük boyutlarda bunun birkaç örneğini inceleyelim.

$V = \mathbb{R}$ 1-boyutlu reel vektör uzayı üzerindeki dejenere kuadratik form $Q(x) = 0$ formudur ve bu forma karşılık gelen Clifford cebri $\theta_1 \in \mathbb{R}$ $Q(\theta_1) = 0$ özelliğinde bir eleman olmak üzere $Cl_{0,0,1} = span\{1, \theta_1\}$ şeklinde yazılabilir, bu aynı zamanda $\mathbb{R}$ üzerindeki $\Lambda(\mathbb{R})$ dış cebridir. Daha açık olarak yazılırsa $Cl_{0,0,1} = \{a1 + b\theta_1 \mid a, b \in \mathbb{R}\}$ olur, buradaki çarpma işlemi $a1 + b\theta_1, a'1 + b'\theta_1$ $Cl_{0,0,1}$ cebrinin elemanları olmak üzere

$$(a1 + b\theta_1) \cdot (a'1 + b'\theta_1) = aa'1 + (ab' + ba')\,\theta_1$$

şeklindedir.

$V = \mathbb{R}^n$ üzerinde $Q(x) = 0$ sıfır kuadratik formu alındığında karşılık gelen Clifford cebrine dış cebir (Grasman cebri) denir ve $Cl_{0,0,n} = \Lambda(\mathbb{R}^n)$ şeklinde gösterilir.

$V = \mathbb{R}^2$ 2-boyutlu reel vektör uzayı üzerinde $Q(x) = x_1^2$ dejenere kuadratik formuna karşılık gelen Clifford cebri $Cl_{1,0,1}$ dir. Bu uzayın Sylvester tabanı $\{e_1, \theta_1\}$ olmak üzere $Cl_{1,0,1} = span\{1, e_1, \theta_1, e_1\theta_1\}$ olur, buna göre herhangi $x\epsilon Cl_{1,0,1}$ elemanı

$$x = x_0.1 + x_1e_1 + x_2\theta_1 + x_3e_1\theta_1 \ (x_0, x_1, x_2, x_3 \in \mathbb{R}\ )$$

şeklinde yazılır. Burada skalerle çarpma ilgili elemanın katsayılarının skalerle çarpılması şeklinde, vektörlerin toplamı da aynı taban elemanına karşılık gelen katsayıların toplamı şeklindedir. Vektörlerin çarpımı ise $y \in Cl_{1,0,1}$ ikinci bir eleman olmak üzere

$$\begin{aligned} x \cdot y &= (x_0.1 + x_1e_1 + x_2\theta_1 + x_3e_1\theta_1) \cdot (y_0.1 + y_1e_1 + y_2\theta_1 + y_3e_1\theta_1) \\ &= (x_0y_0 + x_1y_1)\,.1 + (x_0y_1 + x_1y_0)\,e_1 + (x_0y_2 + x_2y_0 + x_1y_3 - x_3y_1)\,\theta_1 \\ &\quad + (x_0y_3 + x_3y_0 + x_1y_2 - x_2y_1)\,e_1\theta_1 \end{aligned}$$

şeklindedir.

$Cl_{1,0,1}$ dejenere Clifford cebri bir matris cebri değildir. Ancak bir matris cebri içerisine yatırmak mümkündür, bunu şöyle başarabiliriz:

Önce $Cl_{1,0,1}$ cebrini, dejenere olmayan $Cl_{2,1}$ Clifford cebri içine yatıralım, $Cl_{2,1}$ cebrinin de $\mathbb{R}\,(2) \oplus \mathbb{R}\,(2)$ matris cebrine izomorf olduğunu biliyoruz.

$Cl_{1,0,1}$ cebrini de $Cl_{2,1}$ cebri içine gömme dönüşümü taban elemanları üzerinde

$$\begin{aligned} f : Cl_{1,0,1} &\hookrightarrow Cl_{2,1} \\ e_1 &\longmapsto e_1 \\ \theta_1 &\longmapsto e_2 + \varepsilon_1 \end{aligned}$$

şeklinde tanımlıdır. Bu şekilde tanımlanan dönüşüm gerçekten bir cebir homomorfizmasıdır. Toplam ve skalerle çarpımın korunduğu aşikardır. Çarpmanın da korunduğunu gösterebiliriz ve bunu da üreteçler üzerinde göstermek yeterlidir. $f(e_1^2) = f(1) = 1 = [f(e_1)]^2 = e_1^2$ ve

$$\begin{aligned} f(\theta_1^2) &= f(0) = 0 = [f(\theta_1)]^2 = [e_2 + \varepsilon_1]^2 \\ &= e_2^2 + e_2\varepsilon_1 + \varepsilon_1 e_2 + \varepsilon_1^2 = 1 + 0 - 1 = 0 \end{aligned}$$

olur ve f 1-1 dir. Bu şekilde tanımlı f homomorfizması ile bölüm 5 de elde edilen $Cl_{2,1}$ cebrinden $\mathbb{R}(2)\oplus\mathbb{R}(2)$ cebrine giden $\Psi_{2,1}$ izomorfizmasının bileşkesi kullanılırsa

$$\begin{aligned} Cl_{1,0,1} &\hookrightarrow Cl_{2,1} \cong \mathbb{R}(2) \oplus \mathbb{R}(2) \\ e_1 &\longmapsto e_1 = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ \theta_1 &\longmapsto e_2 + \varepsilon_1 = \left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right) \end{aligned}$$

şeklinde 1-1 bir cebir homomorfizması yani bir gömme elde edilmiş olur. ($e_1\theta_1 = \left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right)$ dir.)

Buna göre herhangi bir $x \epsilon Cl_{1,0,1}$ elemanını aldığımızda $x = x_0.1 + x_1 e_1 + x_2\theta_1 + x_3 e_1\theta_1$ olarak yazabiliriz. Buradan da $1, e_1, \theta_1, e_1\theta_1$ değerlerini yerine yazarsak aşağıdakileri elde ederiz.

$$\begin{aligned} x &= x_0.\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) + x_1.\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &\quad + x_2.\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right) + x_3.\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} x_0 + x_2 + x_3 & x_1 - x_2 - x_3 \\ x_1 + x_2 + x_3 & x_0 - x_2 - x_3 \end{bmatrix}, \begin{bmatrix} x_0 - x_2 + x_3 & x_1 - x_2 + x_3 \\ x_1 + x_2 - x_3 & x_0 + x_2 - x_3 \end{bmatrix} \right) \end{aligned}$$

Buna göre $Cl_{1,0,1}$ dejenere Clifford cebiri $\mathbb{R}(2) \oplus \mathbb{R}(2)$ matris cebirinin

$$\left\{ \left(\begin{bmatrix} x_0+x_2+x_3 & x_1-x_2-x_3 \\ x_1+x_2+x_3 & x_0-x_2-x_3 \end{bmatrix}, \begin{bmatrix} x_0-x_2+x_3 & x_1-x_2+x_3 \\ x_1+x_2-x_3 & x_0+x_2-x_3 \end{bmatrix} \right) \middle| x_i \in \mathbb{R}, i = 0, \ldots, 3 \right\}$$

şeklindeki alt cebirine izomorftur.

$V = \mathbb{R}^2$ 2-boyutlu reel vektör uzayı üzerinde $Q(x) = -x_1^2$ dejenere kuadratik formuna karşılık gelen Clifford cebri $Cl_{0,1,1}$ dir. Benzer şekilde bu dejenere Clifford cebrini de uygun bir matris cebiri içerisine gömmek mümkündür. Önce $Cl_{0,1,1}$ cebrini dejenere olmayan $Cl_{1,2}$ cebri içerisine yatıralım. Ayrıca $Cl_{1,2}$ cebrinin de $\mathbb{C}(2)$ matris cebrine izomorf olduğunu biliyoruz. $Cl_{0,1,1}$ cebrini $Cl_{1,2}$ cebri içine gömme dönüşümü taban elemanları üzerinde

$$\begin{array}{ccc} Cl_{0,1,1} & \hookrightarrow & Cl_{1,2} \cong \mathbb{C}(2) \\ \varepsilon_1 & \mapsto & \varepsilon_1 \\ \theta_1 & \mapsto & e_1 + \varepsilon_2 \end{array}$$

şeklinde tanımlıdır. Yukarıdaki örneğe benzer şekilde bu şekilde tanımlanan dönüşümün 1-1 ve cebir homomorfizması olduğu gösterilebilir. Bu şekilde tanımlı homomorfizma ile bölüm 5 de elde edilen $Cl_{1,2}$ cebrinde $\mathbb{C}(2)$ cebrine giden $\Psi_{1,2}$ izomorfizmasının bileşkesi kullanılırsa

$$\begin{array}{ccc} Cl_{0,1,1} & \hookrightarrow & Cl_{1,2} \cong \mathbb{C}(2) \\ \varepsilon_1 & \mapsto & \varepsilon_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \varepsilon_1\theta_1 = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix} \\ \theta_1 & \mapsto & e_1 + \varepsilon_2 = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \end{array}$$

şeklinde bir gömme elde edilmiş olur. Buna göre herhangi bir $x \epsilon Cl_{0,1,1}$ elemanını $x = x_0.1 + x_1\varepsilon_1 + x_2\theta_1 + x_3\varepsilon_1\theta_1$ şeklinde yazabiliriz. Buradan da $1, \varepsilon_1, \theta_1, \varepsilon_1\theta_1$ değerlerini yerine yazarsak aşağıdakileri elde ederiz.

$$\begin{aligned} x &= x_0.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1.\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + x_2.\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} + x_3.\begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix} \\ &= \begin{bmatrix} x_0 - x_3 + x_2 i & -x_1 + x_2 + x_3 i \\ x_1 + x_2 + x_3 i & x_0 + x_3 - x_2 i \end{bmatrix} \end{aligned}$$

Buradan

$$Cl_{0,1,1} \cong \left\{ \left[\begin{array}{cc} x_0 - x_3 + x_2 i & -x_1 + x_2 + x_3 i \\ x_1 + x_2 + x_3 i & x_0 + x_3 - x_2 i \end{array} \right] \middle| x_i \in \mathbb{R}, i = 0,1,2,3 \right\} \subset \mathbb{C}(2)$$

olduğu elde edilir.

$V = \mathbb{R}^3$ 3-boyutlu reel vektör uzayı üzerinde $Q(x) = -x_1^2 - x_2^2$ dejenere kuadratik formuna karşılık gelen Clifford cebri $Cl_{0,2,1}$ dir. Benzer şekilde bu dejenere Clifford cebrini de uygun bir matris cebiri içerisine gömmek mümkündür. Bunu yukarıdaki örneklerde belirttiğimiz gibi $Cl_{0,2,1}$ cebrini önce dejenere olamayan $Cl_{1,3}$ cebri içerisine yatırıyoruz. $Cl_{1,3}$ cebrinin $\mathbb{H}(2)$ matris cebrine izomorf olduğunu biliyoruz. $Cl_{0,2,1}$ cebri de $Cl_{1,3}$ cebri içerisine gömme dönüşümü taban elemanları üzerinde

$$\begin{array}{ccc} Cl_{0,2,1} & \hookrightarrow & Cl_{1,3} \cong \mathbb{H}(2) \\ \varepsilon_1 & \mapsto & \varepsilon_1 \\ \varepsilon_2 & \mapsto & \varepsilon_2 \\ \theta_1 & \mapsto & e_1 + \varepsilon_3 \end{array}$$

şeklinde tanımlıdır. Bu şekilde tanımlı homomorfizma ile bölüm 5 de elde edilen $Cl_{1,3}$ cebrinden $\mathbb{H}(2)$ cebrine giden $\Psi_{1,3}$ izomorfizmasının bileşkesi alınırsa

$$\begin{array}{ccl} Cl_{0,2,1} & \hookrightarrow & Cl_{1,3} \cong \mathbb{H}(2) \\ \varepsilon_1 & \mapsto & \varepsilon_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \varepsilon_2 & \mapsto & \varepsilon_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\ \theta_1 & \mapsto & e_1 + \varepsilon_3 = \begin{bmatrix} j & 1 \\ 1 & -j \end{bmatrix} \end{array}$$

$$\varepsilon_1\varepsilon_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \varepsilon_1\theta_1 = \begin{bmatrix} -1 & j \\ j & 1 \end{bmatrix}$$

$$\varepsilon_2\theta_1 = \begin{bmatrix} k & i \\ -i & k \end{bmatrix}, \quad \varepsilon_1\varepsilon_2\theta_1 = \begin{bmatrix} i & -k \\ k & i \end{bmatrix}$$

şeklinde bir gömme elde edilmiş olur. Buna göre herhangi bir $x \epsilon Cl_{0,2,1}$ elemanını

$$x = x_0.1 + x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\theta_1 + x_4\varepsilon_1\varepsilon_2 + x_5\varepsilon_1\theta_1 + x_6\varepsilon_2\theta_1 + x_7\varepsilon_1\varepsilon_2\theta_1$$

şeklinde yazabiliriz.

Buradan da $\varepsilon_1, \varepsilon_2, \theta_1, \varepsilon_1\varepsilon_2, \varepsilon_1\theta_1, \varepsilon_2\theta_1, \varepsilon_1\varepsilon_2\theta_1$ değerlerini yerine yazarsak aşağıdakiler elde edilir.

$$\begin{aligned} x &= x_0.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1.\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + x_2.\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + x_3.\begin{bmatrix} j & 1 \\ 1 & -j \end{bmatrix} \\ &+ x_4\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + x_5\begin{bmatrix} -1 & j \\ j & 1 \end{bmatrix} + x_6\begin{bmatrix} k & i \\ -i & k \end{bmatrix} + x_7\begin{bmatrix} i & -k \\ k & i \end{bmatrix} \end{aligned}$$

$$x = \begin{bmatrix} x_0 - x_5 + (x_2 + x_7)\,i + x_3 j + x_6 k & -x_1 + x_3 + (x_4 + x_6)\,i + x_5 - x_7 k \\ x_1 + x_3 + (x_4 - x_6)\,i + x_5 j + x_7 k & x_0 + x_5 + (-x_2 + x_7)\,i - x_3 j + x_6 k \end{bmatrix}$$

O halde $Cl_{0,2,1}$ dejenere Clifford cebri $x_i \in \mathbb{R}$ ve $i = 0, ..., 7$ olmak üzere $\mathbb{H}(2)$ matris cebrinin

$$\left\{ \begin{bmatrix} x_0 - x_5 + (x_2 + x_7)\,i + x_3 j + x_6 k & -x_1 + x_3 + (x_4 + x_6)\,i + x_5 - x_7 k \\ x_1 + x_3 + (x_4 - x_6)\,i + x_5 j + x_7 k & x_0 + x_5 + (-x_2 + x_7)\,i - x_3 j + x_6 k \end{bmatrix} \right\}$$

şeklindeki alt cebrine izomorf olur.

$V = \mathbb{R}^3$ 3-boyutlu reel vektör uzayı üzerinde $Q(x) = x_1^2 - x_2^2$ dejenere kuadratik formuna karşılık gelen Clifford cebri $Cl_{1,1,1}$ dir. Benzer şekilde bu dejenere Clifford cebrini de uygun bir matris cebiri içerisine gömmek mümkündür. Bunu yukarıdaki örneklerde belirttiğimiz gibi $Cl_{1,1,1}$ cebrini önce dejenere olamayan $Cl_{2,2}$ cebri içerisine yatırıyoruz. $Cl_{2,2}$ cebrinin $\mathbb{R}(4)$ matris cebrine izomorf olduğunu biliyoruz. $Cl_{1,1,1}$ cebrinin de $Cl_{2,2}$ cebri içerisine gömme dönüşümü taban elemanları üzerinde

$$\begin{array}{ccc} Cl_{1,1,1} & \hookrightarrow & Cl_{2,2} \cong \mathbb{R}(4) \\ e_1 & \mapsto & e_1 \\ \varepsilon_1 & \mapsto & \varepsilon_1 \\ \theta_1 & \mapsto & e_2 + \varepsilon_2 \end{array}$$

şeklinde tanımlıdır.

Bu şekilde tanımlı homomorfizma ile bölüm 5 de elde edilen $Cl_{2,2}$ cebrinden $\mathbb{R}(4)$ cebrine giden $\Psi_{1,3}$ izomorfizmasının bileşkesi alınırsa

$$\begin{array}{lcl} Cl_{1,1,1} & \hookrightarrow & Cl_{2,2} \cong \mathbb{R}(4) \\ e_1 & \longmapsto & e_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \varepsilon_1 & \longmapsto & \varepsilon_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \theta_1 & \longmapsto & \theta_1 = e_2 + \varepsilon_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \end{array}$$

şeklinde bir gömme elde edilmiş olur. Burada

$$e_1\varepsilon_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, e_1\theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix},$$

$$\varepsilon_1\theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}, e_1\varepsilon_1\theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

eşitlikleri vardır. Buna göre herhangi bir $x \epsilon Cl_{1,1,1}$ elemanını

$$x = x_0.1 + x_1e_1 + x_2\varepsilon_1 + x_3\theta_1 + x_4e_1\varepsilon_1 + x_5\varepsilon_1\theta_1 + x_6e_1\theta_1 + x_7e_1\varepsilon_1\theta_1$$

şeklinde yazabiliriz.

Buradan da $\varepsilon_1, \varepsilon_2, \theta_1, \varepsilon_1\varepsilon_2, \varepsilon_1\theta_1, \varepsilon_2\theta_1, \varepsilon_1\varepsilon_2\theta_1$ değerlerini yerine yazarsak aşağıdakileri elde ederiz.

$$x = x_0.\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + x_1.\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + x_2.\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$+x_3.\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} + x_4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + x_5\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

$$+x_6\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} + x_7\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

Buradan

$$x = \begin{bmatrix} x_0 + x_4 & x_1 - x_2 & 0 & 0 \\ x_1 + x_2 & x_0 - x_4 & 0 & 0 \\ 2x_3 + 2x_7 & 2x_5 - 2x_6 & x_0 + x_4 & x_1 - x_2 \\ 2x_5 + 2x_6 & -2x_3 + 2x_7 & x_1 + x_2 & x_0 - x_4 \end{bmatrix}$$

olduğu elde edilir.

O halde $Cl_{1,1,1}$ dejenere Clifford cebri $\mathbb{R}(4)$ matris cebrinin

$$\left\{ \begin{bmatrix} x_0 + x_4 & x_1 - x_2 & 0 & 0 \\ x_1 + x_2 & x_0 - x_4 & 0 & 0 \\ 2x_3 + 2x_7 & 2x_5 - 2x_6 & x_0 + x_4 & x_1 - x_2 \\ 2x_5 + 2x_6 & -2x_3 + 2x_7 & x_1 + x_2 & x_0 - x_4 \end{bmatrix} \middle| \, x_i \in \mathbb{R}, i = 0, 1, ..., 7 \right\}$$

şeklindeki alt cebrine izomorf olur.

Aşağıda bazı dejenere Clifford cebirlerinin ilgili matris cebri içerisine gömme dönüşümleri verilmiştir.

Yukarıda yaptığımız örneklere benzer şekilde gömme dönüşümlerinin açık ifadeleri elde edilebilir.

$$\begin{array}{lcl}
Cl_{0,3,1} & \hookrightarrow & Cl_{1,4} \cong \mathbb{H}(2) \oplus \mathbb{H}(2) \\
Cl_{0,4,1} & \hookrightarrow & Cl_{1,5} \cong \mathbb{H}(4) \\
Cl_{0,1,2} & \hookrightarrow & Cl_{2,3} \cong \mathbb{C}(4) \\
Cl_{0,2,2} & \hookrightarrow & Cl_{2,4} \cong \mathbb{H}(4) \\
Cl_{0,3,2} & \hookrightarrow & Cl_{2,5} \cong \mathbb{H}(4) \oplus \mathbb{H}(4) \\
Cl_{1,1,1} & \hookrightarrow & Cl_{2,2} \cong \mathbb{R}(4) \\
Cl_{1,2,1} & \hookrightarrow & Cl_{2,3} \cong \mathbb{C}(4) \\
Cl_{1,3,1} & \hookrightarrow & Cl_{2,4} \cong \mathbb{H}(4) \\
Cl_{1,4,1} & \hookrightarrow & Cl_{2,5} \cong \mathbb{H}(4) \oplus \mathbb{H}(4) \\
Cl_{1,0,2} & \hookrightarrow & Cl_{3,2} \cong \mathbb{R}(4) \oplus \mathbb{R}(4)
\end{array}$$

Buradan tüm $Cl_{p,q,r}$ dejenere Clifford cebirleri için sözkonusu gömmelerin ne olduğunu aşağıdaki tablodan yararlanarak kolaylıkla bulabiliriz.

$(p-q) \pmod 8$	$p+q$	$Cl_{p,q,r}$ nin içine gömüldüğü matris cebri
$0, 2$	$2m$	$\mathbb{R}(2^{m+r})$
1	$2m+1$	$\mathbb{R}(2^{m+r}) \oplus \mathbb{R}(2^{m+r})$
$3, 7$	$2m+1$	$\mathbb{C}(2^{m+r})$
$4, 6$	$2m+2$	$\mathbb{H}(2^{m+r})$
5	$2m+3$	$\mathbb{H}(2^{m+r}) \oplus \mathbb{H}(2^{m+r})$

Şimdi yukarıdaki örneklerin ışığı altında dejenere Clifford cebirleri ile ilgili bir teorem ifade edelim:

Teorem 8.1 *$Cl_{p,q,r}$ dejenere Clifford cebri, dejenere olmayan $Cl_{p+r,q+r}$ Clifford cebri içerisine gömülebilir.*

Kanıt. Bunun için $Cl_{p,q,r}$ den $Cl_{p+r,q+r}$ ye giden 1-1 bir cebir homomorfizmi vermek yeterlidir. Söz konusu homomorfizmi de taban elemanları üzerinde tanımlayıp tüm cebire genişletmek mümkündür.

Taban elemanları üzerinde aşağıdaki şekilde tanımlanan dönüşüm istenilen homomorfizmi verir:

$$\begin{array}{rcl} \Psi_{p,q,r}: \; Cl_{p,q,r} & \hookrightarrow & Cl_{p+r,q+r} \\ e_1 & \mapsto & e_1 \\ \vdots & & \vdots \\ e_p & \mapsto & e_p \\ \varepsilon_1 & \mapsto & \varepsilon_1 \\ \vdots & & \vdots \\ \varepsilon_q & \mapsto & \varepsilon_q \\ \theta_1 & \mapsto & e_{p+1} + \varepsilon_{q+1} \\ \vdots & & \vdots \\ \theta_r & \mapsto & e_{p+r} + \varepsilon_{q+r} \end{array}$$

Gerçekten

$$\Psi_{p,q,r}(e_i^2) = \Psi_{p,q,r}(1) = 1 = [\Psi_{p,q,r}(e_i)]^2 = e_i^2$$

$$\Psi_{p,q,r}(\varepsilon_i^2) = \Psi_{p,q,r}(1) = -1 = [\Psi_{p,q,r}(\varepsilon_i)]^2 = \varepsilon_i^2$$

$$\begin{aligned} \Psi_{p,q,r}(\theta_i^2) &= \Psi_{p,q,r}(0) = 0 = [\Psi_{p,q,r}(\theta_i)]^2 = [e_{p+i} + \varepsilon_{q+i}]^2 \\ &= e_{p+i}^2 + e_{p+i}\varepsilon_{q+i} + \varepsilon_{q+i}e_{p+i} + \varepsilon_{q+i}^2 \\ &= 1 + e_{p+i}\varepsilon_{q+i} - e_{p+i}\varepsilon_{q+i} - 1 = 0 \end{aligned}$$

eşitlikleri vardır. ■

Bu teoremden aşağıdaki sonuç elde edilir.

Sonuç 8.1 *$Cl_{p,q,r}$ dejenere Clifford cebri $\mathbb{R}(m), \mathbb{C}(m)$, $\mathbb{H}(m)$, $\mathbb{R}(m) \oplus \mathbb{R}(m)$, $\mathbb{H}(m) \oplus \mathbb{H}(m)$ tipindeki matris cebirlerinin içine gömülebilir.*

8.2 Dejenere Clifford cebirlerinin reel temsilleri

Bölüm 7 da dejenere olmayan reel Clifford cebirlerinin reel temsillerinin açık ifadelerini elde etmiştik. Dejenere durumdaki reel Clifford cebirlerinin reel temsillerini, dejenere olmayan reel Clifford cebirlerinin reel temsilleri yardımıyla elde etmek mümkündür.

Bunu da önce örnekler üzerinden inceleyelim.

$Cl_{0,1,1}$ dejenere Clifford cebrini dejenere olmayan $Cl_{1,2}$ cebrinin içerisine gömüldüğünü yukarıda söyledik. Şimdi $Cl_{1,2}$ nin Bölüm 7 de verilen reel temsili yardımıyla $Cl_{0,1,1}$ in bir reel temsilini elde edelim. Hatırlanacağı üzere bir A reel cebrinin reel temsilini vermek demek A dan uygun bir $\mathbb{R}(m)$ ye giden bir cebir homomorfizmi vermek demektir. A nın bir Clifford cebri olması durumunda ise söz konusu homomorfizmin üreteçler üzerindeki değerlerini bilmek yeterli olur. Örneğimize geri dönecek olursak: Öncelikle

$$\begin{array}{lcl} Cl_{0,1,1} & \hookrightarrow & Cl_{1,2} \\ \varepsilon_1 & \mapsto & \varepsilon_1 \\ \theta_1 & \mapsto & e_1 + \varepsilon_2 \end{array}$$

gömmesiyle $Cl_{0,1,1}$ den $Cl_{1,2}$ ye gidilirse ve sonrada Bölüm 7 de $Cl_{1,2}$ nin verilen reel temsili kullanılırsa

$$\begin{array}{lcl} Cl_{0,1,1} & \to & \mathbb{R}(4) \\ \varepsilon_1 & \mapsto & \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = I \otimes \sigma_1\sigma_2 \\ \theta_1 & \mapsto & \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} = I \otimes \sigma_1 + \sigma_1\sigma_2 \otimes \sigma_2 \end{array}$$

elde edilir.

$Cl_{1,0,1}$ dejenere Clifford cebrini dejenere olmayan $Cl_{2,1}$ cebrinin içerisine gömüldüğünü yukarıda söyledik. Şimdi $Cl_{2,1}$ nin Bölüm 7 de verilen reel temsili yardımıyla $Cl_{1,0,1}$ in bir reel temsilini elde edelim.

Öncelikle

$$\begin{array}{lcl} Cl_{1,0,1} & \hookrightarrow & Cl_{2,1} \\ e_1 & \mapsto & e_1 \\ \theta_1 & \mapsto & e_2 + \varepsilon_1 \end{array}$$

gömmesiyle $Cl_{1,0,1}$ den $Cl_{2,1}$ e gidilip ve sonra da Bölüm 7 de $Cl_{2,1}$ nin verilen reel temsili kullanılırsa

$$\begin{array}{lcl} Cl_{1,0,1} & \rightarrow & \mathbb{R}(2) \\ e_1 & \longmapsto & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \theta_1 & \longmapsto & \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \end{array}$$

$Cl_{1,0,1}$ dejenere Clifford cebrinin reel temsili elde edilir.

Şimdi daha genel olarak $Cl_{p,q,r}$ dejenere Clifford cebirlerinin reel temsillerini vereceğiz. Bunu verirken de Bölüm 7 de verilen dejenere olmayan Clifford cebirlerinin reel temsillerini kullanacağız.

$$\begin{array}{ccccc} Cl_{p,q,r} & \hookrightarrow & Cl_{p+r,q+r} & \rightarrow & \mathbb{R}(n) \\ e_1 & \mapsto & e_1 & \mapsto & A_1 \\ \vdots & & \vdots & & \vdots \\ e_p & \mapsto & e_p & \mapsto & A_p \\ \varepsilon_1 & \mapsto & \varepsilon_1 & \mapsto & B_1 \\ \vdots & & \vdots & & \vdots \\ \varepsilon_q & \mapsto & \varepsilon_q & \mapsto & B_q \\ \theta_1 & \mapsto & e_{p+1} + \varepsilon_{q+1} & \mapsto & A_{p+1} + B_{q+1} \\ \vdots & & \vdots & & \vdots \\ \theta_r & \mapsto & e_{p+r} + \varepsilon_{q+r} & \mapsto & A_{p+r} + B_{q+r} \end{array}$$

Burada $1 \leq i \leq p + r$ ve $1 \leq j \leq q + r$ olmak üzere A_i ve B_j ler dejenere olmayan $Cl_{p+r,q+r}$ Clifford cebrinin reel temsilidir. Ayrıca herhangi farklı iki taban elemanı x ve y ise $xy = -yx$ özelliği vardır.

Bu durumda $e_i^2 = 1, \varepsilon_j^2 = -1$ ve $1 \leq k \leq r$ olmak üzere,

$$\begin{aligned} (e_{p+k} + \varepsilon_{q+k})^2 &= e_{p+k}^2 + e_{p+k}\varepsilon_{q+k} + \varepsilon_{q+k}e_{p+k} + \varepsilon_{q+k}^2 \\ &= 1 + 0 - 1 = 0 \end{aligned}$$

dır.

Not: Yukarıda elde edilen temsiller indirgenemez değildir. Örneğin $Cl_{1,0,1}$ in yukarıda elde edilen temsiline ρ diyelim. Herhangi bir $x \in Cl_{1,0,1}$ elemanını $x = x_0 1 + x_1 e_1 + x_2 \theta_1 + x_3 e_1 \theta_1$ şeklinde yazabiliriz. O zaman

$$\begin{aligned} \rho(x) &= x_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} x_0 + x_2 + x_3 & x_1 - x_2 - x_3 \\ x_1 + x_2 + x_3 & x_0 - x_2 - x_3 \end{bmatrix} \end{aligned}$$

olur. $\mathbb{R}^2$ nin $V = \{(v, v) \mid v \in \mathbb{R}\}$ alt uzayını aldığımızda $\rho(x) V \subset V$ olduğu elde edilir. Gerçekten

$$\rho(x) V = \begin{bmatrix} x_0 + x_2 + x_3 & x_1 - x_2 - x_3 \\ x_1 + x_2 + x_3 & x_0 - x_2 - x_3 \end{bmatrix} \begin{bmatrix} v \\ v \end{bmatrix} = \begin{bmatrix} (x_0 + x_1) v \\ (x_0 + x_1) v \end{bmatrix}$$

olur. O halde $Cl_{1,0,1}$ dejenere Clifford cebrinin temsili indirgenemez değildir.

Bunu genel durumda da aşağıdaki teoremin sonucu olarak da söyleyebiliriz (Bakınız [3]).

Teorem 8.2 *A reel bir cebir ve $\rho : A \to \mathbb{R}(m)$ indirgenemez bir temsil ise, $Rad(A) = N$ olmak üzere, her $a \in N$ için $\rho(a) = 0$ dır.*

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