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Research Article / Araştırma Makalesi

Determination of the Relative Positions of Three Planes: Action Research

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Abstract

The purpose of this study was to explore how a more effective lesson plan and teaching environment can be achieved so as to improve elementary mathematics teacher candidates′ achievement in analytical examination of planes in space. In order to improve achievement in expressing the relative positions of three planes not only algebraically but also visually the study used an action research approach as planned by the researchers. In Implementation 1, the teacher candidates were given the equations of three planes and they were asked to determine the relative positions of the planes so that their prior knowledge could be identified. In this stage, the candidate teachers tried to determine the relative positions of the planes in one direction by examining the plane equations in pairs. In Implementation 2, the candidate teachers were asked to find the solution set of the linear equation system consisting of three equations with three unknowns and to come up with geometric interpretation of this solution. In this stage, some of the candidate teachers were able to solve the equation, but they couldn′t interpret it geometrically. In Implementation 3, Maple, a computer algebra system, was used so that the candidate teachers could visualize and observe the relative positions of the three planes by using the plane equations. In this stage, the candidate teachers associated the set of solutions of the plane equations with the three-dimensional images obtained with Maple. The

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results of the implementation showed that the proposed plan improved the mathematics teacher candidates' visualization of the relative positions of the three planes.

Keywords: *planes in space, analytic geometry, Maple, action research*

Üç Düzlemin Birbirine Göre Konumunun Belirlenmesi: Eylem Araştırması

Öz

Bu çalışmanın amacı, ilköğretim Matematik öğretmen adaylarının uzayda düzlemlerin analitik incelenmesi konusundaki başarısını arttırmak için daha etkili bir ders planının ve öğretim ortamının nasıl sağlanacağını araştırmaktır. Bu amaca ulaşabilmek için araştırmacıların hazırladığı plan dahilinde eylem araştırması yaklaşımı kullanılmıştır. Uygulamanın 1. aşamasında öğretmen adaylarına üç düzlemin denklemleri verilerek düzlemlerin birbirine göre konumlarının belirlenmesi istenmiştir. Bu aşamada öğretmen adayları düzlemlerin birbirine göre konumunu ikişer ikişer düzlem denklemlerini inceleyerek tek yönlü belirlemeye çalışmıştır. Uygulamanın ikinci aşamasında üç bilinmeyenli üç denklemden oluşan lineer denklem sistemlerinin çözümünü bulmaları ve bu çözümü geometrik olarak yorumlamaları istenmiştir. Bu aşamada ise öğretmen adaylarının bir kısmı denklem sistemini çözdüğü ancak geometrik olarak yorumlayamadığı gözlemlenmiştir. Üçüncü aşamada ise üç düzlemin birbirine göre konumunun görselleştirmesi ve düzlem denklemlerini kullanarak bu düzlemlerin birbirlerine göre konumlarını gözlemleyebilmeleri için bilgisayar cebir sistemlerinden Maple programı kullanılmıştır. Bu aşamada öğretmen adayları düzlem denklemlerinin oluşturduğu denklem sistemlerinin çözüm kümesi ile Maple da elde edilen üç boyutlu görselleri ilişkilendirmişlerdir. Uygulamadan elde edilen bulgulara göre hazırlanan planın matematik öğretmen adaylarının üç düzlemin birbirine göre konumunun görselleştirmesinde olumlu etkisi olmuştur.

Anahtar kelimeler: *Uzayda düzlemler, analitik geometri, Maple, eylem araştırması*

Introduction

Plane geometry consists of basic structures called points and lines. On the other hand, solid geometry includes basic structures called planes in addition to points and lines. Plane geometry deals with planar geometric structures with all the elements located in the same plane. However, solid geometry involves an infinite number of planes. It is important to explore the relative positions of these planes in space. This examination could be both synthetic and algebraic with the help of plane equations. In algebraic terms, the plane is represented by a linear equation with three unknowns, $Ax + By + Cz + D = 0$, $A, B, C, D \in R$. Therefore, determining the relative positions of three planes requires examining the solution of a linear system of equations consisting of three equations with three unknowns. The geometric interpretation of this solution set of planes allows determining the relative positions of the planes. Geometric interpretation of the positions of planes in space requires three-dimensional thinking. According to our personal observations during classes as teachers, students often have difficulty in making sense of the relative positions of planes in space. This is probably because students have difficulty in visualizing the relative positions of three planes since three-dimensional thinking skills in analytical geometry are not developed in most students (Schumann, 2003). Bako (2003) suggested that problems in teaching the subjects of space geometry exist mainly because students cannot see in three dimensions. On the other hand, research showed that students have difficulty in comprehending spatial states when they are taught three-dimensional objects using traditional tools in the classroom environment such as pencil and paper (Baki, Kösa and Karakuş, 2008; Kösa, Karakuş and Çakıroğlu, 2008). Therefore, selecting the right method for teaching spatial states in space geometry could help students mentally visualize objects in three dimensions. In fact, teachers could both present the subjects of space geometry, which are difficult to display and require considerable repetition and drawing practice, both more easily and more effectively by using the right technological tools in the classroom (Kösa, Karakuş and Çakıroğlu, 2008). Computer Algebra Systems (CAS) such as Maple, Mathematica, Derive and so on are used in analytic geometry courses because they allow for performing algebra operations and spatial visualization thanks to their three-dimensional graph drawing functions. In this study, Maple was used for visualization the relative positions of planes in three-dimensional analytic geometry.

An effective lesson plan that is compatible with the course objectives both helps the teacher present the subject and help students comprehend the subject better. It is essential that teachers use appropriate methods and strategies to prepare lesson plans. A teacher can investigate the impact of the lesson plan with an action research in which he or she develops action plans in response to the questions “How can I present the subject most efficiently?” and “How can I prepare the most appropriate lesson plan?” (Baki, Erkan and Demir, 2012).

The purpose of this study was to explore how a more effective lesson plan and teaching environment can be achieved so as to improve elementary mathematics teacher candidates’ achievement in analytical examination of planes in space. In the light of this, we developed a lesson plan and investigated its efficacy.

Methodology

In our study, we used the action research method in which the teacher also acted as a researcher. In this method, the teacher develops a solution for a problem encountered in the classroom and implements it in order to improve the quality of teaching in the learning environment (Çepni, 2010). Both of the authors of our study currently teach Analytic Geometry at two different universities. Each of us identified the misconceptions of teacher candidates about the subjects in planes in space and noted down the difficulties encountered by teacher candidates during algebraic and analytical examination of the relative positions of planes in space. By means of the action research proposed in this study, we developed and implemented activities to eliminate these misconceptions. As a part of the action research cycle in this study, the implementation was explored through planning, acting, reflecting and evaluation phases (Çepni, 2010).

Participants

This study enrolled a total of 42 third year students studying Elementary Mathematics Education at a state university in Turkey. A lesson plan was prepared and this plan was implemented for five hours of class with these students who were taking Analytical Geometry at the time of this research. In order to keep secret the identity of the participants, S1, S2, ... coding is used.

Action Research Process

The action research process started with the diagnosing phase and it continued with the planning, acting and evaluation phases. These phases are given below in order.

Diagnosing phase

Based on our previous teaching experience and personal observations in Space Analytic Geometry courses at our own universities, we identified the problem situation about “determining the positions of three planes in space”. We came together to come up with ideas about a more effective teaching of the subject ‘planes in space’ and to prepare a plan about the subject. The diagnosing phase was followed by the planning phase in which an answer to the question “How can we improve achievement in not only algebraic but also visual representation of the relative positions of planes in space?”

Planning phase

We prepared a three-stage plan. The first stage included structured questions involving various situations in response to the question “What are participants’ ways of thinking in determining the relative positions of three given planes?” in order to determine participants’ prior knowledge about the subject. The second stage consisted of structured questions including various cases in order to determine their use of systems of linear equations in determining the relative positions of planes. Finally, the third stage included activities that were planned using Computer Algebra Systems (CAS) so that the set of solutions of the linear equation systems of the plane equations could be interpreted geometrically and the relative positions of the planes could be visualized.

Activity1

The aim of this activity was to examine the relative positions of three parallel planes analytically. The participants studied a relevant sample.

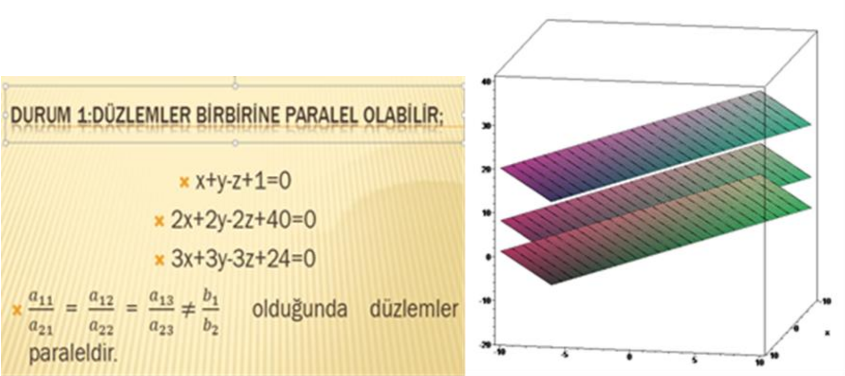


Figure 1. Equations and graphs of parallel planes

Activity 2

The aim of this activity was to examine the relative positions of three congruent planes analytically. The participants studied a relevant sample.

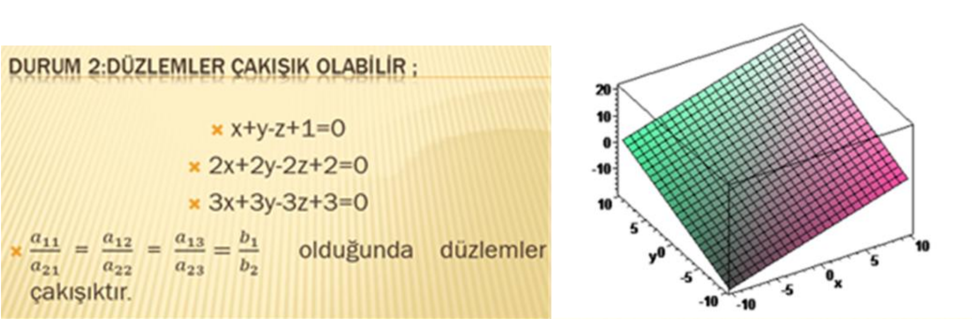


Figure 2. Equations and graphs of cutting planes

Activity 3

This activity was prepared to analytically examine the relative positions of three planes two of which are parallel and one of which intersects each of them along straight lines. The participants studied a relevant sample.

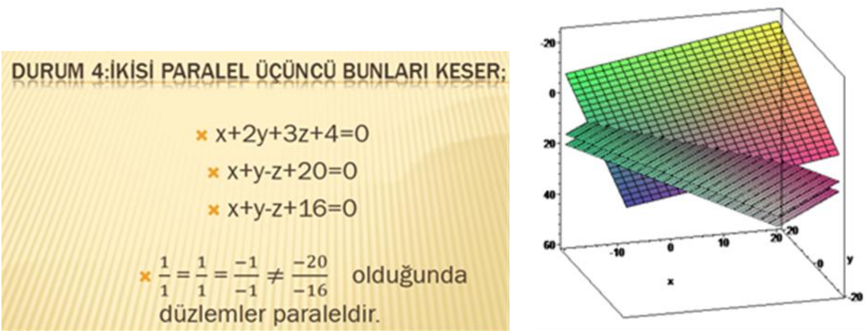


Figure 3. Equations and graphs of planes two of which are parallel and one of which intersects each of them along straight lines

Activity 4

This activity was prepared to analytically examine the relative positions of three planes that intersect in pairs along straight lines. The participants examined a relevant sample.

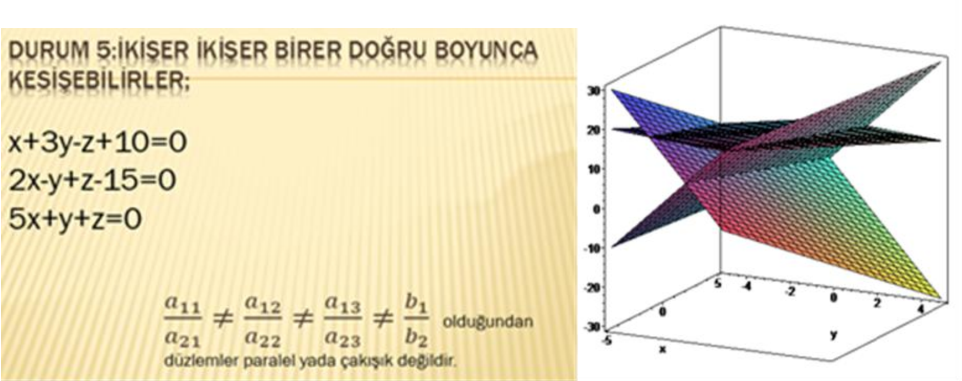


Figure 4. Equations and graphs of planes that intersect along in pairs a straight line

Activity 5

The aim of this activity was to analytically examine the relative positions of three planes that intersect along a straight line. The participants studied a relevant sample.

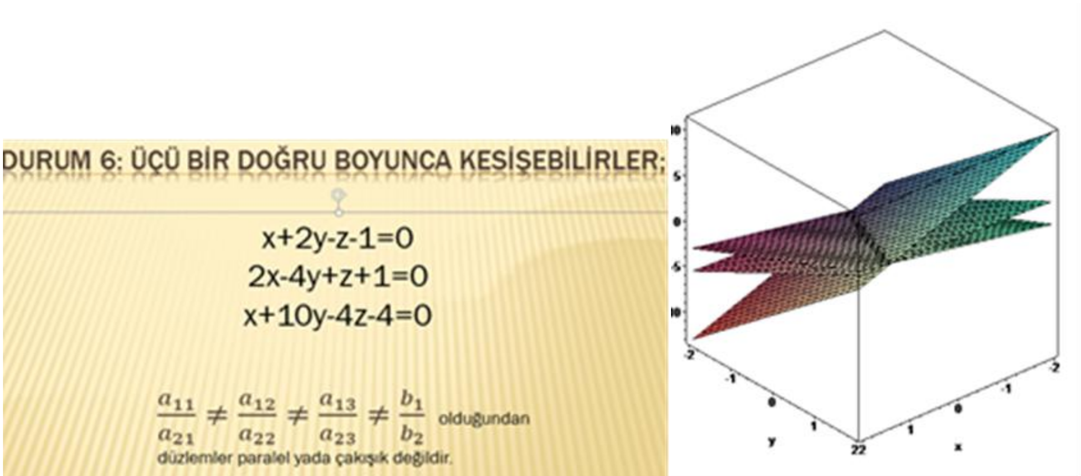


Figure 5. Equations and graphs of planes that intersect along a straight line

Activity 6

The aim of this activity was to analytically examine the relative positions of three planes that intersect at a single point. The participants studied a relevant sample.

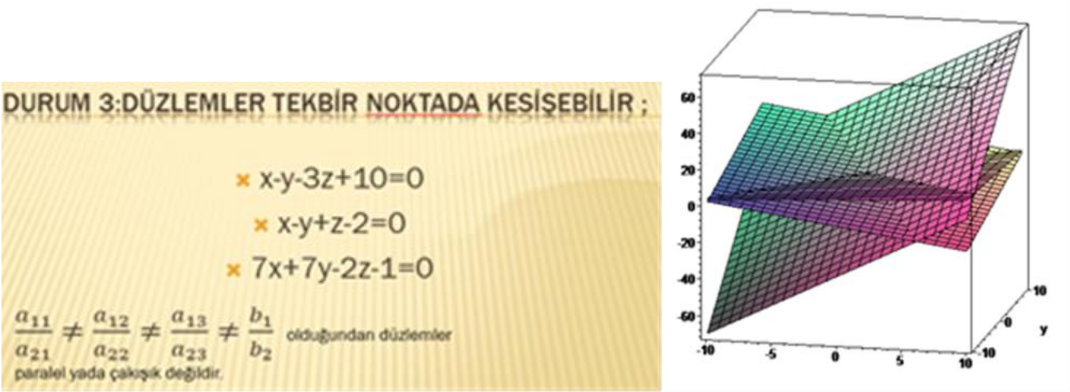


Figure 6. Equations and graphs of planes that intersect at a single point

Acting phase

In the light of the proposed plan, the lesson consisted of three stages. One of the researchers acted as the teacher in the classroom while the other researcher video-recorded the implementation process, the participants' responses and the role of the teacher in the classroom. The implementations were conducted in a classroom equipped with an interactive whiteboard. The activities in the implementation were also shared with the participants on the interactive whiteboard. In addition, the observer-researcher adopted a critical eye toward the possible drawbacks of the plan throughout the implementation.

Evaluation phase

In this phase of the study, we came together and examined the notes taken, video recordings, the responses to the structured questions. In this phase, we sought an answer to the question "What is the impact of the prepared plan on student learning?" At the end of this phase, we reached a consensus that the activities served their intended goals, the plan prepared was accepted as an effective plan, and this process was reported.

Data Analysis

Data were collected using structured questions and the observation technique. Theme were identified from the observations and activities and data were interpreted. Data obtained from the structured questions were analyzed both quantitatively and qualitatively by using the descriptive data analysis. The descriptive approach was used because the purpose of this study was to describe and explain all the details of the case in question. The descriptive analysis included sample authentic extracts from the participant papers in order to reflect the participants' ways of thinking clearly.

Findings

This study followed the research process described in the previous section, and the lesson was conducted in the light of the activities designed to eliminate errors made in determining the relative positions of planes in space. The results from the three-stage implementation steps with identified themes are as follows, respectively.

Implementation 1

Theme 1: The participants' prior knowledge

This stage of the implementation focused on determining the participants' prior knowledge about the subject by identifying their ways of thinking in determining the relative positions of three given planes. Therefore, Test 1 was developed. This test included structured questions involving the following states: "three parallel planes, three planes intersecting along a straight line, and three planes intersecting at a single point". The test was administered to 42 teacher candidates who participated in the implementation. Table 1 shows the results and frequency table.

Table 1

Frequency Table Representing the Participants' Prior Knowledge about the Subject

TEST 1	Question 1 (Parallel Planes)	Question 2 (Intersect at a single point)	Question 3 (Intersect in pairs along straight lines)
True	15	12	-
False	27	29	41
Empty	-	1	1

As can be seen in Table 1, Questions 1 and 2 were answered correctly by nearly 35% though none of the participants were able to give a correct answer to Question 3. Analysis of the answer sheets showed that the participants who gave correct answers determined the relative positions of the planes by using the vectors perpendicular to planes, which are referred to as the normals to a plane (i.e. x, y and z coefficients in the plane equation respectively), or by using the determinant of coefficient matrix of equations.

Those participants who used the determinant of coefficient matrix commented that the planes would be parallel if the determinant was equal to zero but they would intersect at a point if the determinant was not equal to zero. They could not, however, correctly answer Question 3, in which the determinant of coefficient matrix was equal to zero but the planes were not parallel. In general, according to the results from Test 1, about 70% of the participants failed in determining the relative positions of the planes.

One of the participants, S1, examined the relative positions of the planes by using the normals to the given planes. While she was able to easily determine the planes the normals of which were parallel, like the case in Question 1, she examined the determinant of coefficient matrix for the planes x+2y-z-1=0, 2x-4y+z+1=0 and x+10y-4z-4=0, the normals of which were not parallel, like the case in Question 3, and she considered the determinant being equal to zero as the linear dependence of normal vectors. On the other hand, she was unable to interpret the relative positions of planes (see Figure 7).

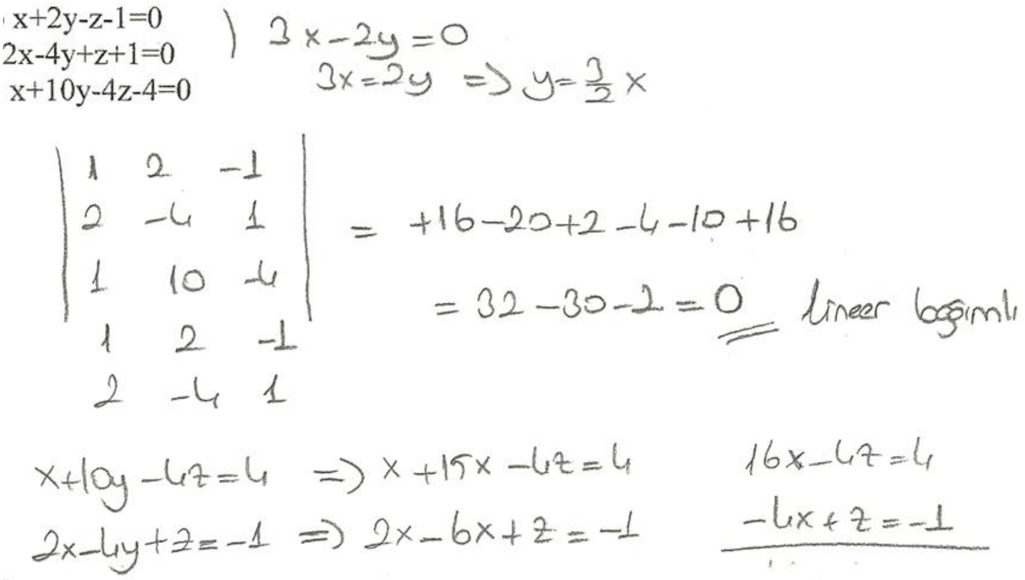


Figure 7. A sample participant answer to Question 3

In fact, in this case (i.e. Question 3), the three planes did not have a single common point but they intersected along a straight line (see Figure 8). It is crucial that participants mentally visualize and make sense of this position.

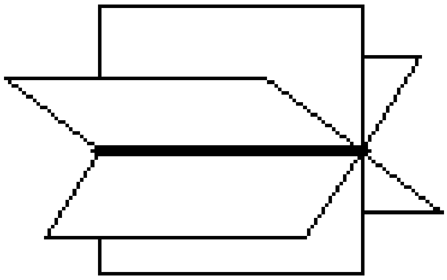


Figure 8. The model for three planes intersecting along a straight line

Another participant, S2, compared the normals to the given planes in pairs and commented that the planes would be parallel if the normals were parallel (see Figure 9).

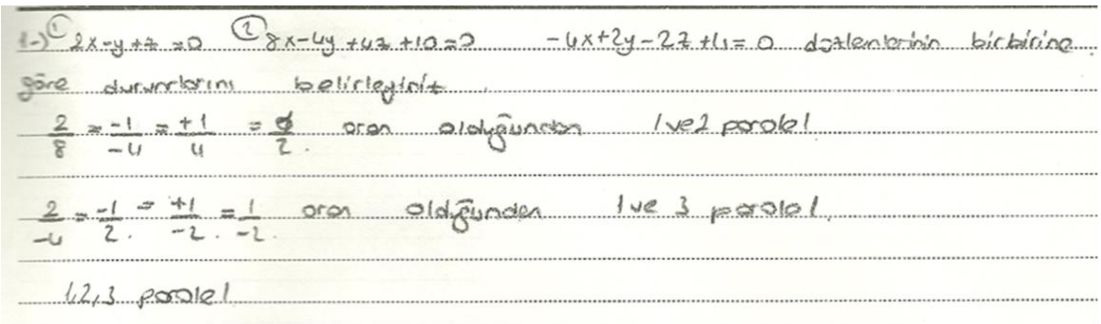


Figure 9. A sample participant answer to Question 1

Then the same participant considered the given plane equations to be a system of linear equations if the normals were not parallel and examined the determinant of coefficient matrix. Finally, she concluded that the single point, which she found by solving the system of equations if the determinant was not equal to zero, would be the intersecting point of the planes (see Figure 10).

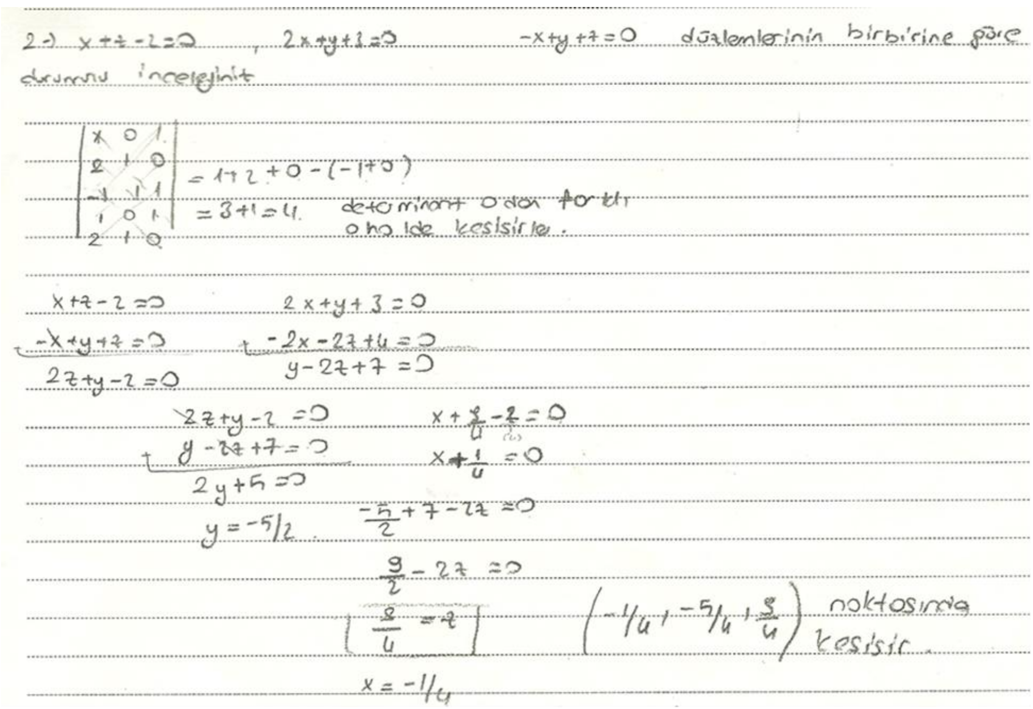


Figure 10. A sample participant answer to Question 2

On the other hand, in Question 3, in which the normals were not parallel and the determinant of coefficient matrix was equal to zero, the same participant solved the equations of the planes x+2y-z-1=0, 2x-4y+z+1=0 and x+10y-4z-4=0 like a system of equations and concluded that the system would have infinite solutions. Nevertheless, she was not able to interpret this correct solution as three planes intersecting along a straight line (see Figure 11).

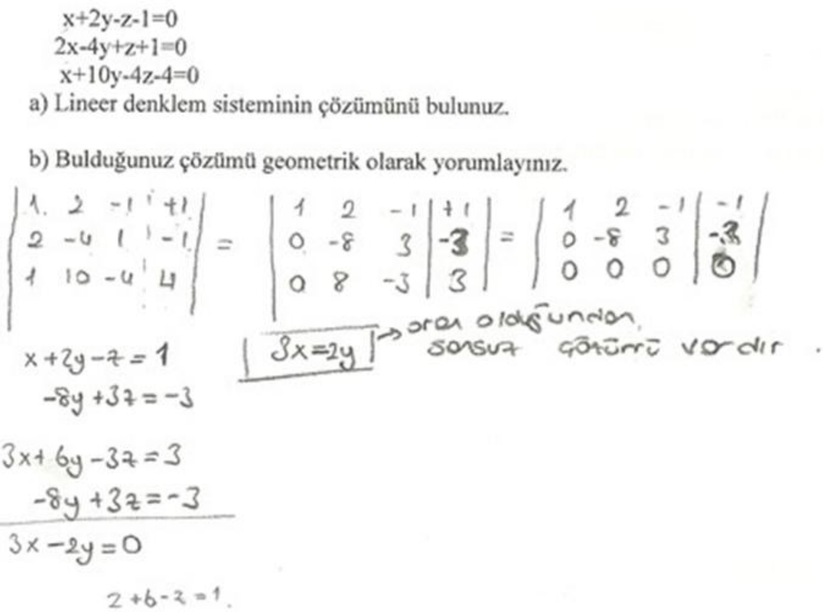


Figure 11. A sample participant answer to Question 3

Clearly, using only normals or only the determinant of the coefficient matrix is not enough for determining the relative positions of planes. In order to address this problem, therefore, Test 2 was developed. This test required the participants to find and interpret the solution set of the linear equation systems consisting of three equations with three unknowns.

Implementation 2

Theme 1: The participants' use of linear equation systems

In the second stage of the implementation, Test 2 was developed in order to determine the participants' use of linear equation systems in determining the relative positions of planes. Test 2 consisted of structured questions regarding the following states: "three parallel planes, three planes intersecting at a single point, and three planes intersecting in pairs along straight lines". In fact, the participants were given the plane equations representing the three planes algebraically as systems of linear equations and they were asked to find the set of solutions of this equation system and make a geometrical interpretation. The 42 teacher candidates who participated in the implementation were administered the test. Table 2 shows the results and frequency table.

Table 2

Frequency Table Representing the Teacher Candidates' Use of Linear Equation Systems about the Subject

TEST 2		Question1 (Parallel Lines)	Question 2 (Intersect at a single point)	Question 3 (Intersect in pairs along straight lines)
Algebraic	True	2	27	4
	False	15	8	16
	Empty	25	7	22
Geometric	True	16	18	-
	False	14	10	28
	Empty	12	14	14

As can be seen in Table 2, while there were two participants who gave the correct algebraic solution by solving the equation system in Question 1, 16 participants came up with the correct geometric interpretation of this algebraic solution. The answer sheets of those participants who did not provide an algebraic solution but made a correct geometric interpretation showed that they realized that the equations forming the system of equations were actually the algebraic equation of a plane and they noted that the planes were parallel to each other by using vectors, referred to as the normals to planes, without solving the system of equations or by using the determinant of coefficient matrix. As can be seen in Figure 12, S2 examined the determinants of coefficient matrix without finding the solution set of the equation system. S2 further commented that if the determinant was equal to zero, the three planes would be parallel.

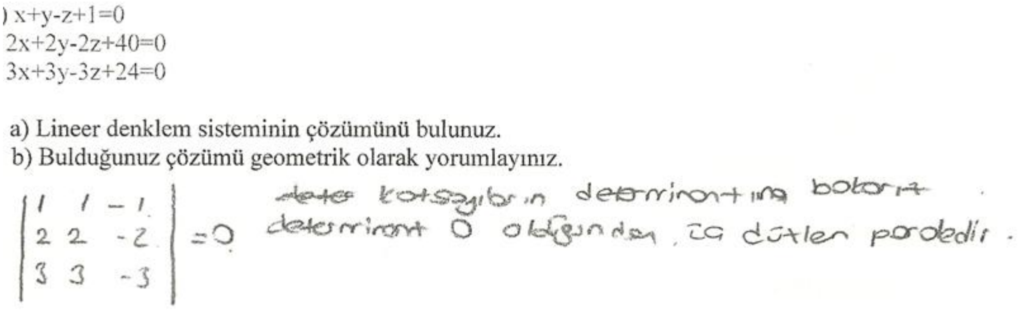


Figure 12. A sample algebraic and geometric participant (S2) answer to Question 1

In this question (Question 1), the three planes are truly parallel, but since the determinant value would be equal to zero even if two of the planes were parallel while the third plane intersect each of them along straight lines, interpreting the relative positions of the planes based only on the determinant value would lead to misconceptions. Indeed, S3 calculated the determinant of the coefficient matrix of the equation system in both Question 1, in which the planes were parallel, and in Question 3, in which the planes intersected each other in pairs along straight lines and she found the correct results of both, zero (see Figure 13 and Figure 14).

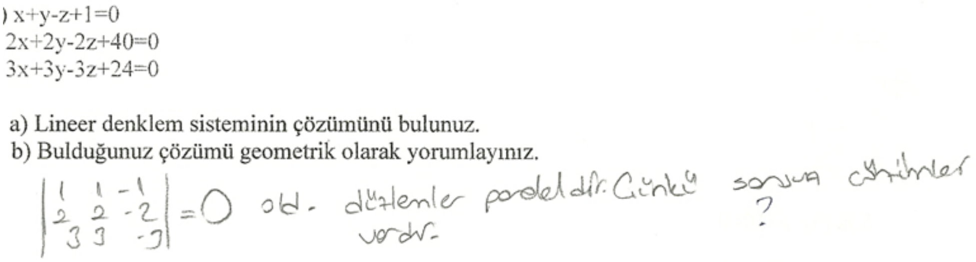
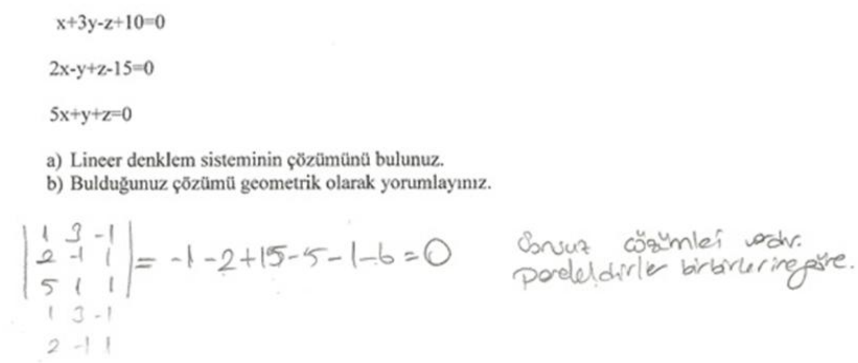


Figure 13. Sample algebraic and geometric participant (S3) answers to Question 1


x+3y-z+10=0

2x-y+z-15=0

5x+y+z=0

a) Lineer denklem sisteminin çözümünü bulunuz.
b) Bulduğunuz çözümü geometrik olarak yorumlayınız.

$\begin{vmatrix} 1 & 3 & -1 \\ 2 & -1 & 1 \\ 5 & 1 & 1 \end{vmatrix} = -1-2+15-5-1-6=0$

Sonsuz çözümleri vardır.
Paraleldirler birbirlerine göre.

Figure 14. Sample algebraic and geometric participant (S3) answers to Question 3

In both of her answers given in Figure 13 and Figure 14, without solving the equation system, S3 commented, “The system has infinite solutions. Then the planes must be parallel to each other” (see Figure 14). However, if the determinant was equal to zero, the solution set of the system of equations would be a null set,

- In Figure 15, the three planes could be parallel to each other,

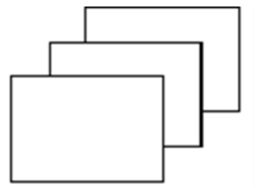


Figure 15. Three parallel planes

- In Figure 16, two of the planes could be parallel while the third plane could intersect each one along straight lines,

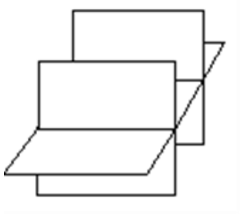


Figure 16. Two parallel planes intersected by a third one

- In Figure 17, the planes could intersect in pairs along straight lines.

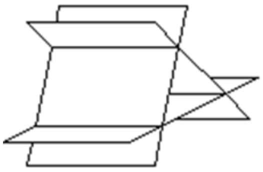


Figure 17. Planes intersecting in pairs

In Question 2, 27 participants provided correct algebraic answers and 18 of them also made correct interpretations while 9 did not provide any answers at all. The numbers of both algebraically and geometrically correct answers given by the participants for this question were higher then the correct answers given to the other questions mainly because the participants generalized that a single point in the algebraic solution would be the common point of the planes. This shows that the participants made inferences based on the answers. They also generalized that there would not be any common points if the solution set was a null set, but they did not come up with any responses about determination of the relative positions of the planes. In addition, when they encountered a parametric solution in the solution set of the equation system most of the participants except for few of them were not able make a geometric interpretation.

Among the 42 participants, just four gave the correct algebraic answer to Question 3. In other words, four participants solved the linear equation system correctly and found the solution set as a null set. However, although only two of them geometrically interpreted this case, they wrote down “the planes do not intersect,” “these equations do not intersect as they do not have a common solution.” (see Figure 18).

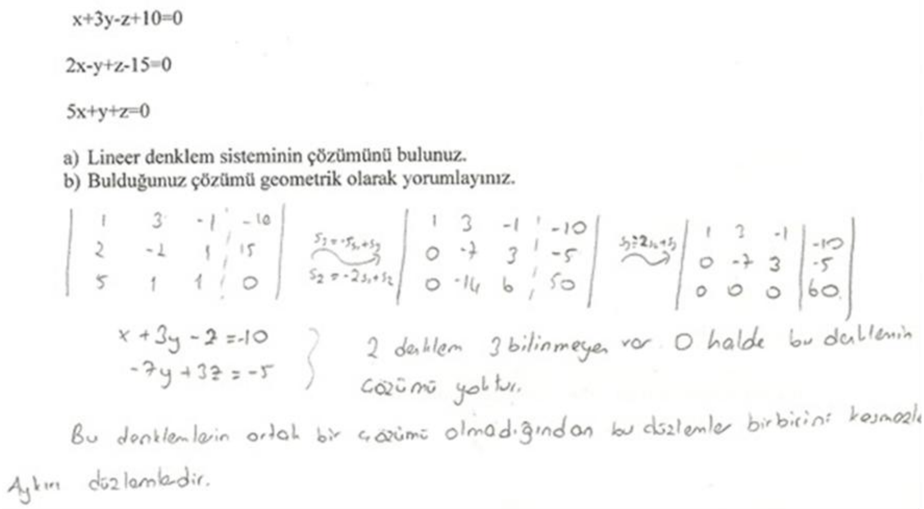


Figure 18. A sample algebraic and geometric participant answer to Question 3

The three planes do not have a common point. However, if we examine them in pairs, we could realize that the planes intersect along straight lines. The second participant's explanation that these planes would not intersect is a proof of the misconception of the participant. The other two participants who found the correct algebraic solution to the question, on the other hand, did not provide any geometric interpretation.

In Test 2, although the participants were given the plane equations as an equation system and required to find the solution set of the system initially, those participants who gave answers using the normals to the planes in Test 1 preferred making an interpretation using the normals to the planes again without finding the solution set of the system.

Those participants who mentioned the parallelism of the planes by using the parallelism of the normals as the given planes for Question 1 in Test 2, which were parallel to each other, did not use this strategy for Question 3, in which the planes intersect in pairs and, therefore, they were not able to provide a correct geometric interpretation. For instance, S1 interpreted the parallelism of the planes due to the parallelism of the normals. Nevertheless, when the planes were not parallel, she found the determinant of the coefficient matrix of the system, she considered the determinant being equal to zero as the linear dependence of the normal vectors (see Figure 19), she considered the determinant not being equal to zero as the linear independence of the normal vectors, but she did not provide any interpretation about the relative positions of the planes (see Figure 20).

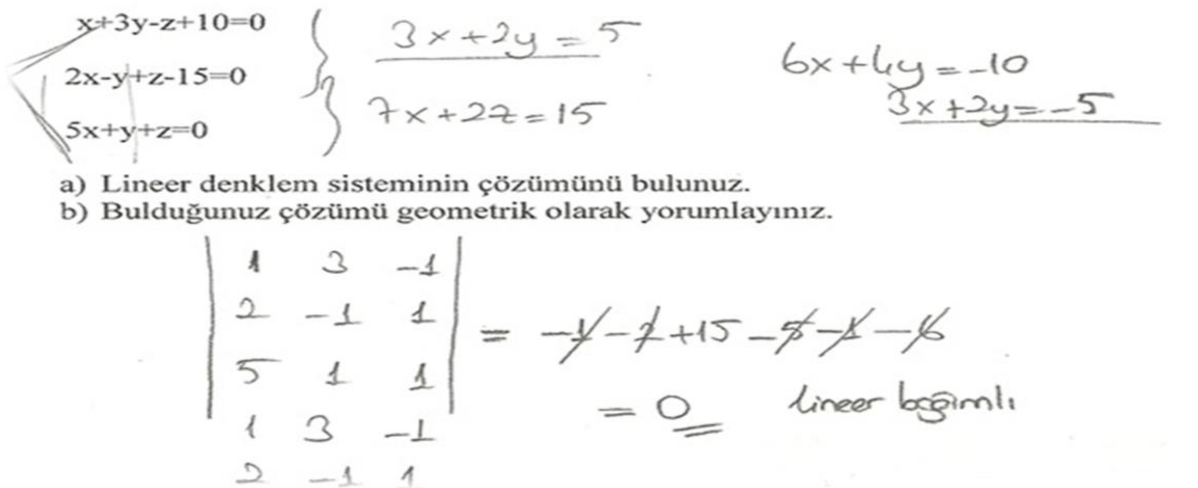


Figure 19. A sample algebraic and geometric participant answer to Question 5

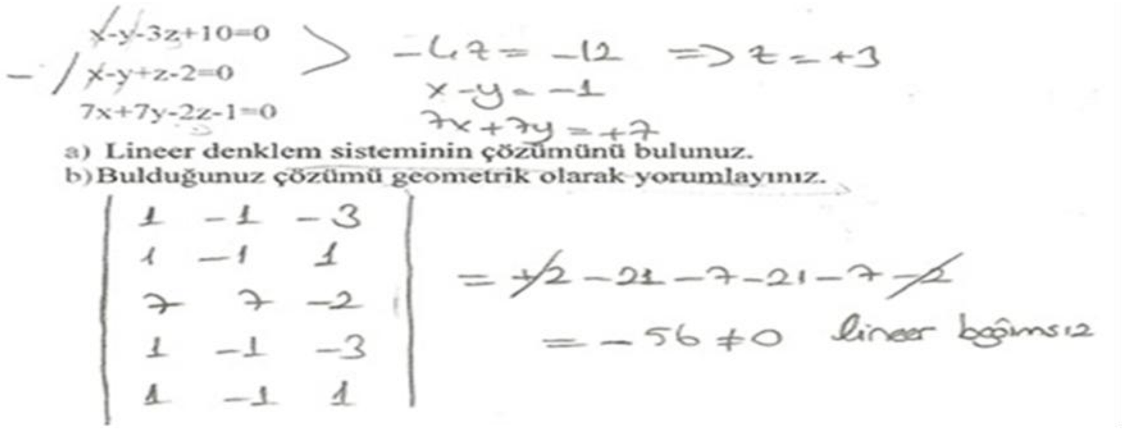


Figure 20. A sample algebraic and geometric participant answer to Question 3

In addition, she did not find the solution set of the equation system for any question of this test. That prevented the same participant from interpreting the planes with non-parallel normals.

According to the analysis of Test 1 and Test 2 together, the achievement of the participants was 35% when the three planes were parallel or when the three of them intersected at a single point whereas none of the participants were able to determine the cases when the three planes intersected along a single straight line or the planes intersected each other in pairs along straight lines.

We observed that this situation was caused by the fact that they did not use the equation system to determine the common point of these planes or they had difficulty in finding the solution set of the equation system or they failed in providing an appropriate geometric interpretation of the solution set they found. In order to eliminate this problem, Implementation 3 was planned based on the following steps:

1. performing the algebraic solution of the equation system formed by the plane equations,
2. use of this algebraic solution in determining the relative positions of the planes,
3. geometric visualization of the position of the graphs drawn with a computer program for determining the position of planes.

Implementation 3

Theme 3: Visualization of the planes with CAS

In this stage of implementation, many sample exercises were practiced so as to increase the comprehension of the activities performed so far, and all of the participants were required to participate in these activities. In this way, a common environment for participation and discussion emerged within the classroom, and the class discussed their answers and errors as a whole. The teacher-researcher encouraged the participants to share their answers in the classroom, provided feedback, eliminated misconceptions about the subject and had the participants revise the subject. In this sense, the teacher-researcher contributed to achieving the purpose of the implementation. We also observed that the use of CAS for visualizing the subject increased the participants’ interest. The contents of the activities carried out in the lesson are described below in detail.

Since the participants were required to determine the relative positions of the three planes that were parallel to each other in Activity 1 and the relative positions of the three planes that intersected each other in Activity 2, they were able to reach the correct solution by using the equations of the given planes and by using the normals to the planes in both of the activities. Those participants who used the determinant of the coefficient matrix of the equation system formed by the plane equations and who found it to be equal to zero in these two activities were clearly explained that the reason for this situation was the parallelism of the normal vectors of the planes. These cases were visualized with Maple.

On the other hand, the participants found the determinant of the coefficient matrix of the given equation systems to be equal to zero in Activity 3, in which two of the planes were parallel and the third plane intersected each of them along straight lines; in Activity 4, in which the three planes intersected in pairs along straight lines; and in Activity 5, in which the three planes intersected a straight line. The participants were then required to find the solution sets of equations. In this case, they noticed that the solution sets in Activity 3 and Activity 4 were null sets. They were then asked to examine the planes in pairs because the three planes did not have a common point. In this case, the solution set of the equation system was a null set when the determinant was equal to zero. The participants realized that the three planes could be parallel to each other or they could intersect each other, two of the planes could be parallel while the third one could intersect each of them along straight lines, or the planes could intersect in pairs along straight lines. Each of these cases was visualized with Maple.

In Activity 5, the participants found the solution set of the linear equation system formed by the equations of the planes to be infinite solution that depends on a parameter. Since this situation corresponds to geometrically collinear infinite points, the participants concluded that the three planes intersected along a straight line. This case was also visualized with Maple.

In Activity 6, the participants found the determinant of the coefficient matrix of the equation system obtained from the three planes given initially to be unequal to zero. Using Cramer's rule, they found the solution set of the equation system as a single point. It was by means of Maple that the participants identified this point as the cut-off point of the three planes.

Like the case with Implementation 1 and Implementation 2, the vast majority of the participants either calculated the determinant of the coefficient matrix of the linear equation system formed by the equations of the planes or used the normals to the planes when determining the relative positions of the planes (see Figure 21).

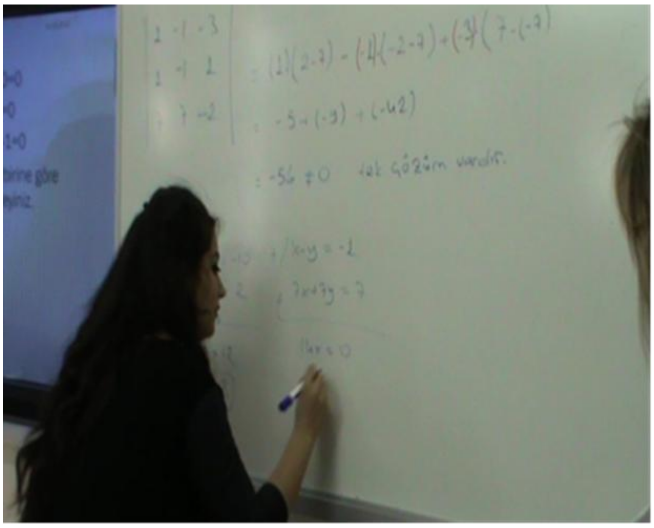


Figure 21. A participant performing solution at board

Practicing various positions of the planes presented in Activity 1-Activity 6 helped the participants realize that their previous strategies were inadequate. We observed that visualization of geometric representation of the solution sets obtained with CAS in Implementation 3 facilitated the participants' interpretation of the solution set. Moreover, visualization of the relative positions of the planes with CAS for each different case helped the participants mentally visualize the three-dimensional geometric state.

Finally, the results from Test 3, which was developed to determine the participants' achievement in determining the relative positions of planes in space showed that they solved the linear equation system formed by the equations of the planes and then they interpreted this solution geometrically. On the other hand, in Test 3, some of the participants produced visual representations of the relative positions of the planes with their drawings. This test included questions involving the following states: "planes intersecting with each other in pairs along straight lines, planes intersecting along a single straight line and two parallel planes each intersected by a third one along straight lines". A total of 42 participants were administered the test. Table 3 shows the results and frequency table.

Table 3

Frequency Table Representing the Participants' Achievement In Determining The Relative Positions Of Planes

Test 3	Question 1 (intersect in pairs along straight lines)	Question 2 (intersect at a single point)	Question 3 (two parallel planes each intersected by a third one along straight lines)
True	35	38	33
False	6	4	8
Empty	1	-	1

As can be seen in Table 3, nearly 88% of the participants gave correct answers to Questions1, 2 and 3. The participants' answer sheets showed that they realized they could encounter different cases in determining the relative positions of the planes if the determinant of the coefficient matrix was equal to zero. In fact, we observed that the participants realized that if the determinant of the coefficient matrix of the linear equation system formed by the equations of the planes turned out to be equal to zero, they could encounter three different cases: the planes could be parallel to each other, two of the planes could be parallel to each other while the third one could intersect each of them along straight lines, and the three planes could intersect each other in pairs along straight lines.

In Test 3, a total of 35 participants gave correct answers to Question 1 while six of them gave incorrect answers and one participant did not answer the question. Those participants who gave correct answers solved the linear equation system formed by the plane equations, provided correct geometric interpretations and reached the correct solution. Below is a sample answer sheet of one of the participants, S4 (see Figure 22)

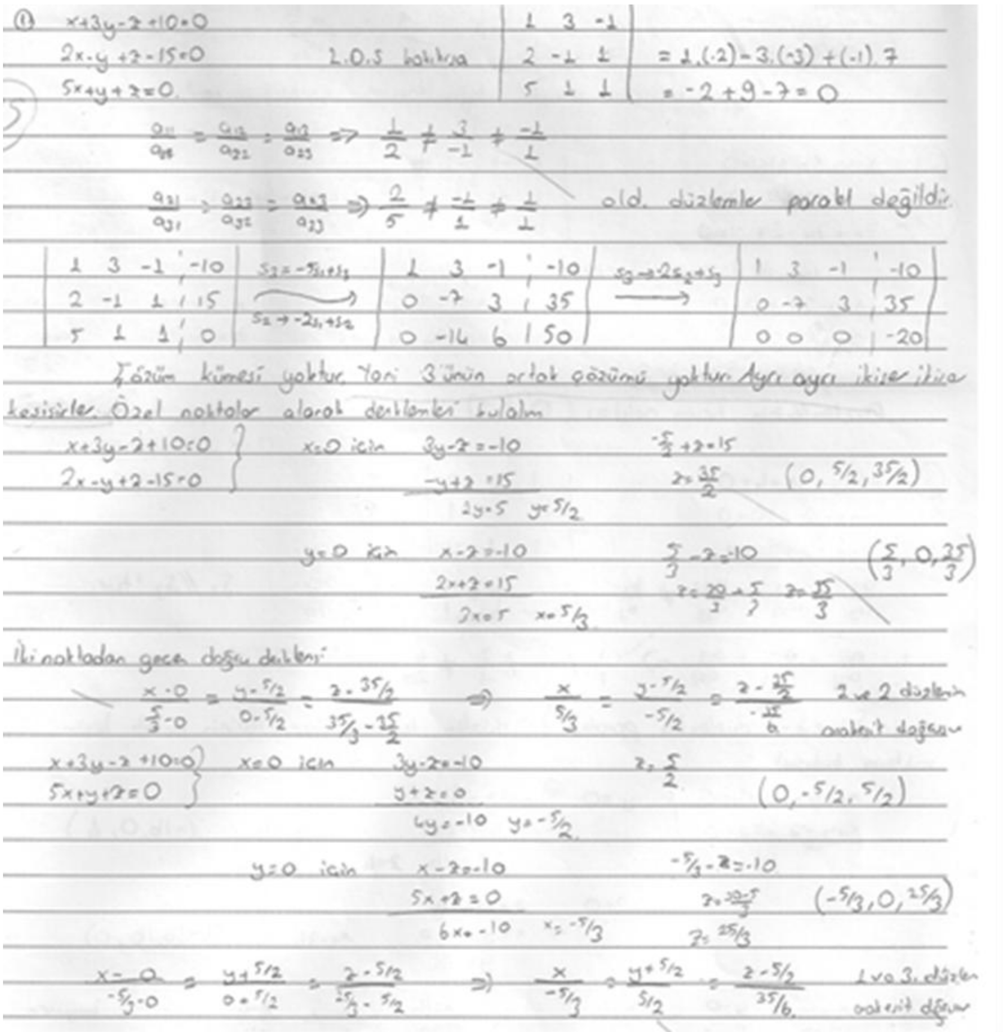


Figure 22. A sample participant answer to Question 1

S4's answer to this question clearly shows that she took the possible different positions of the planes into consideration when the determinant of the coefficient matrix was equal to zero while determining the relative positions of the three planes. In this regard, S4 examined the parallelism of the planes by comparing and contrasting their normals. When she noticed that the planes were not parallel, she reached the conclusion that there was not any solution set by using the elementary row operations of the linear equation system formed by the planes. Based on this conclusion, she came up with the geometric interpretation that the three planes did not have a common point. The same participant then examined the three planes in pairs and determined that the planes intersected each other in pairs along straight parallel lines (see Figure 22).

Most of the six participants who gave incorrect answers to Question 1 still examined the determinant of the coefficient matrix only, and they stated that there would be only coincidence or parallelism if the determinant was equal to zero. These participants made this error because they did not find the solution of the linear equation system formed by the planes, one of the three steps in Implementation 3. For example, S5 found the determinant of the coefficient matrix to be equal to zero but then she stated that the plane could be coincident or parallel (see Figure 23).

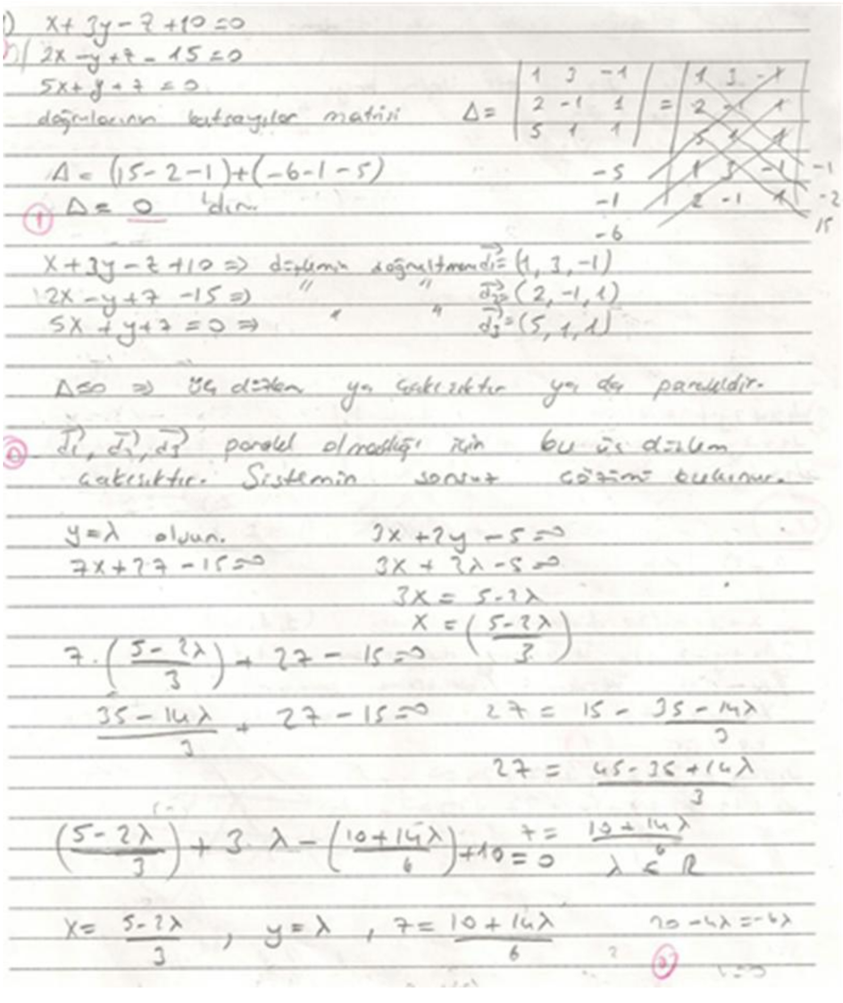


Figure 23. A sample participant answer to Question 1

Comparing and contrasting the normals to the planes, S5 stated that the planes were not parallel but coincident. That was a misinterpretation because she did not solve the linear equation system.

Among the 42 participants, 38 answered Question 2 correctly while four of them gave an incorrect answer. In comparison with the number of participants who gave a correct answer to this question in Test 1, the number of participants who gave a correct answer to the question about the solution of the linear equation system formed by the planes in Test 3 increased dramatically. S3, one of the participants who answered this question correctly, found the determinant of the coefficient matrix of the linear equation system to be unequal to zero and stated that this equation system had a single solution. She then found the cross-section point of the three planes by solving the plane equation in pairs (see Figure 24).

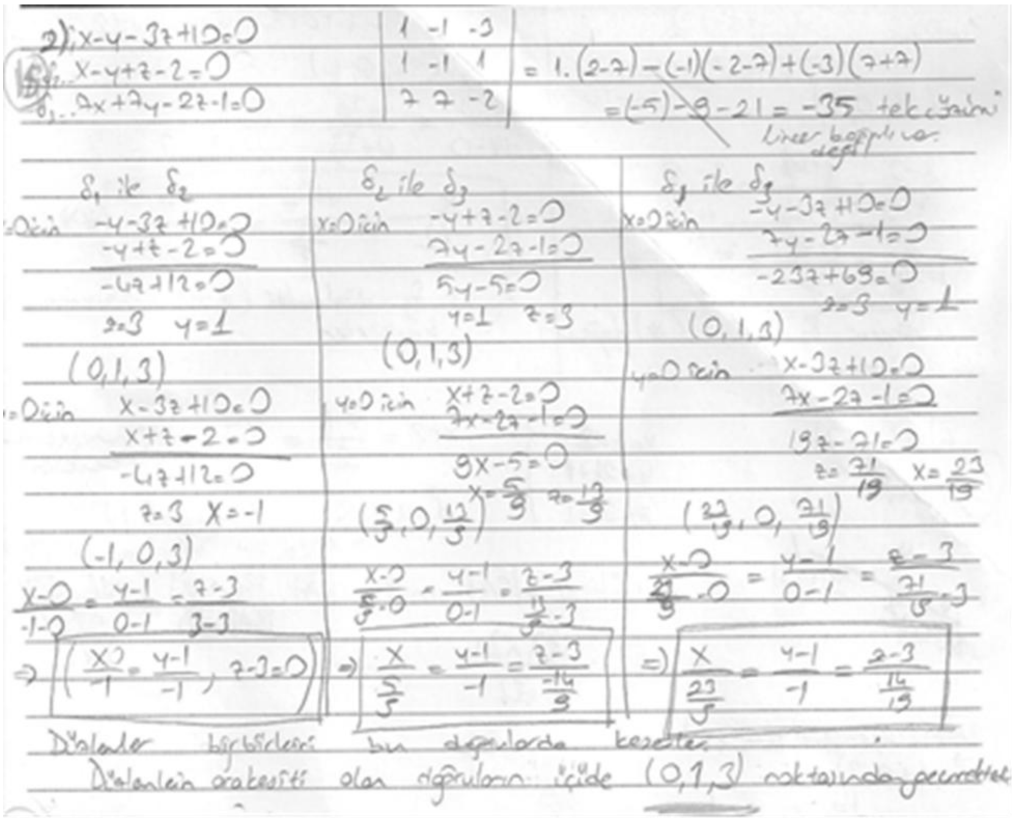


Figure 24. A sample participant answer to Question 2

When she examined the proportion of the normal vector of the planes, on the other hand, she stated that these proportions were not equal to each other and, therefore, the planes were not parallel or coincident. However, her geometric interpretation was missing because she did not solve the linear equation system (see Figure 25).

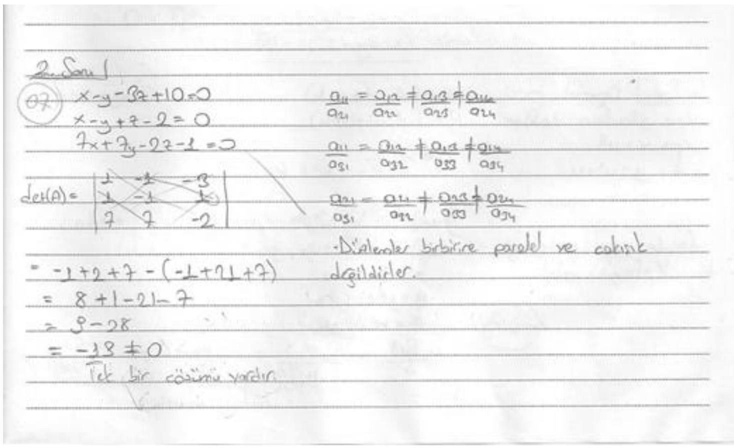


Figure 25. A sample participant answer to Question 2

Among the participants, 33 answered Question 3 in the last test correctly, eight gave an incorrect answer, and one participant did not answer it. In this question, two of the planes were parallel while the third one intersected each of them in pairs along straight lines. By examining the proportions of the normals to the planes in pairs, S6 one of the participants who answered this question correctly, stated that Planes 2 and 3 were parallel to each other but Plane 1 intersected Planes 2 and 3 along straight lines. She also visualized the relative positions of the planes by drawing them (see Figure 26).

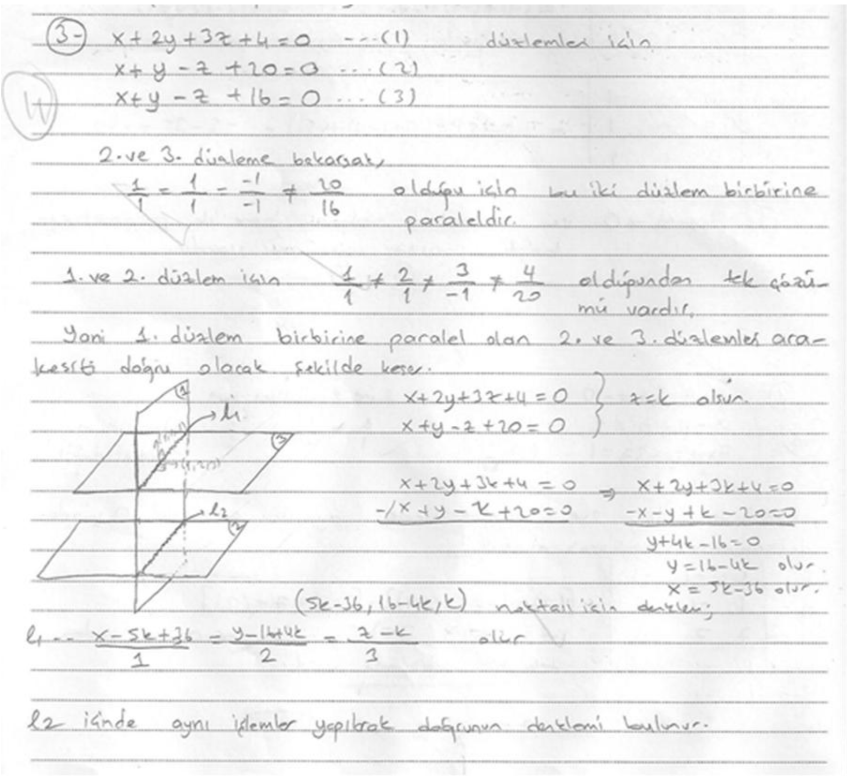


Figure 26. A sample participant answer to Question 3

On the other hand, most of the eight participants who gave an incorrect answer to Question 3 still thought that the determinant of the coefficient matrix turning out to be equal to zero just corresponded to parallelism or coincidence. Although some of these participants found the value of the determinant correctly and came up with a correct geometric interpretation when they made the proportions of the normals to the planes in pairs, they could not find the solution set of the linear equation system. S7 was one of the participants who answered the question in this way (see Figure 27).

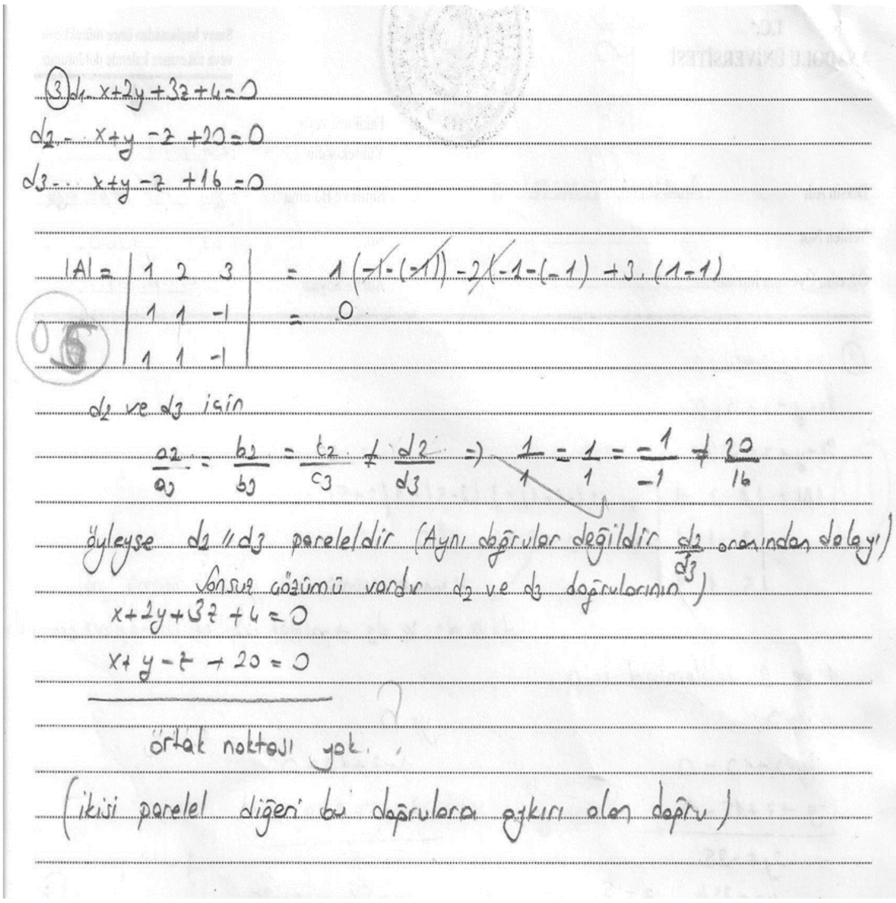


Figure 27. A sample participant answer to Question 3

Although S7 demonstrated the parallelism of the two planes, she wrote that the third plane was transversal to the other two planes. Clearly, her use of the concept of transverse lines not on the same plane was wrong here. Therefore, she could not accurately determine the relationship of the third plane with the two other planes because she did not examine the planes in pairs.

An overall evaluation of the results from Implementations 1, 2 and 3 showed that 88% of the participants were successful at the end of the action plan according to the results from Test 3, which was administered to the participants to determine their achievement in examining the relative positions of the planes analytically. When compared to the results from Test 1, this result indicates that most of the participants overcame their misconceptions about the subject. The implementation steps helped the participants realize that the infinite set of solutions of the system, which the majority of them were not able to interpret before, was actually a solution that depended on a single parameter and this parametric solution was the parametric equation of the cutting line of the planes. Seeing the cutting line of the equations of the planes on screen with a computer software allowed the participants to combine the algebraic result with the geometric visual and, therefore, to make sense of this knowledge. In addition, thanks to the implementation, they realized that even if the equation system formed by the equations of the planes did not have a solution (i.e. if the solution set was a null set), the given three planes did not need to be parallel only but the planes could intersect each other in pairs along straight lines. To sum up, the computer software, CAS, helped the participants reach the correct solution by combining the algebraic solution with its geometric representation.

Discussion and Conclusion

The purpose of this study was to explore how a more effective lesson plan and teaching environment can be achieved so as to improve elementary mathematics teacher candidates' achievement in analytical examination of planes in space. The results showed that the proposed plan improved the participants' achievement in determining the relative positions of the given planes by using the algebraic equations of the planes.

According to the results obtained from the implementation phases of the action plan developed for this study, the participants' ways of thinking about the subject were diagnosed first. Most of the participants just used either the normals to the planes or the coefficient matrix of the equation system formed by the equations of the planes. However, those participants who used this way of solution were not able to obtain a solution in some cases. For example, the participants failed to identify the case in which the three planes intersected along a straight line by using their methods (see Test 1). In this way, the participants' existing misconceptions were identified in Test 1. In the light of this, we needed to help the participants notice the relationship between the plane equations and the linear equations of the first degree with three unknowns. Therefore, in the second implementation, the participants were required to find the solution set of the linear equation system formed by the plane equations and geometrically interpret this solution. Some of the participants were still unable to establish a relationship with the plane equations and linear equation and use the same way of solution they used in the first implementation. The participants were still unable to obtain a result in some cases. For instance, the methods used by the participants did not work in the case when the planes intersect in pairs along straight lines (see Test 2). As shown in Table 2, most of the participants could not find the solution set of the linear equation system. Those who found the solution set, on the other hand, were unable to interpret the result geometrically. Implementation 3 included various activities reflecting all the cases (e.g. a null set as the solution set, an infinite set of solutions depending on a single point and parameter) in which the set of solution of the linear equation system formed by the plane equations was found and the solution was interpreted geometrically under the guidance of the researcher. The solution sets found in these activities were visualized with CAS, and this contributed to eliminating the participants' misconceptions in Test 1 and Test 2. The participants got the opportunity to make observations about the relative positions of the three planes by rotating them at 360 degrees in each direction with CAS. Thus, they were able to see the relative positions and the cross-sections of the planes in space from different angles. The problems encountered in three-dimensional geometry in interpreting the algebraic solution of the equation system were resolved thanks to the dynamic computer program.

The results from Test 3, which was administered to determine the impact of the action plan, showed that the participants were now able to find the solution set of the equation system while examining the relative positions of the three planes for all the questions such as the cases in which the planes intersected each other in pairs along straight lines and two of the planes were parallel to each other while the third one intersected each of them along straight lines. They also accurately interpreted this set of solution geometrically. We could therefore suggest that the prepared action plan achieved its goal.

On the other hand, the results also showed that geometrically interpreting the solution set of the linear equation system formed by the plane equations or examining the planes in pairs turned out to be essential so that the misconceptions of the participants who gave incorrect answers to the questions in Test 3 could be eliminated.

Suggestions

The implementation could be conducted in a computer lab and, therefore, the efficiency of the implementation could be improved by giving every student an opportunity to study with computer software.

References

Baki, A., Kösa, T., & Karakuş, F. (2008). *Uzay geometri öğretiminde 3D dinamik geometri yazılımı kullanımı: Öğretmen Görüşleri*. Paper presented at 8th International Education Technology Conference, Anadolu University, Eskişehir, Turkey.

Baki, A., Erkan, İ., Demir, E.(2012). *Ders planı etkililiğinin lesson study ile geliştirilmesi: bir aksiyon araştırması*. Paper presented at X. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi [X. Congress on International Science and Mathematics Education], Niğde University, Niğde, Turkey.

Bako, M. (2003). *Different projecting methods in teaching spatial geometry*. Retrieved from http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG7/TG7_Bako_cerme3.pdf

Çepni S. (2010). *Araştırma ve proje çalışmalarına giriş*. Trabzon: KTÜ Fatih Eğitim Fakültesi Yayınları.

Kösa, T., Karakuş, F., & Çakıroğlu, Ü. (2008). *Uzay geometri öğretimi için üç boyutlu dinamik geometri yazılımı kullanarak çalışma yapraklarının geliştirilmesi*, Paper presented al International Educational Technology Conference. Anadolu University, Eskişehir, Turkey.

Schumann, H. (2003). Computer aided treatment of 3d- problems in analytic geometry. *Zentralblatt für Didaktik der Mathematik* (*ZDM)*, 35 (1), 7-13.

Genişletilmiş Öz

Bu çalışmanın amacı, ilköğretim Matematik öğretmen adaylarının uzayda düzlemlerin analitik incelenmesi konusundaki başarısını arttırmak için daha etkili bir ders planının ve öğretim ortamının nasıl sağlanacağını araştırmaktır. Bu amaca ulaşabilmek için araştırmacıların hazırladığı plan dahilinde eylem araştırması yaklaşımı kullanılmıştır. Bu yaklaşımın seçilme nedeni, öğretmenin sınıfındaki öğretimin kalitesini arttırmak için öğretim ortamında ortaya çıkan bir probleme çözüm geliştirmesi ve bunu uygulamasıdır (Çepni, 2010). Araştırmacıların her ikisi de farklı üniversitelerde uzay Analitik Geometri dersini yürütmektedir. Araştırmacıların kendi sınıflarında, uzayda düzlem ile ilgili konuları işlerken matematik öğretmen adaylarının yanılgılarını tespit etmiş ve öğrencilerin düzlemlerin birbirine göre konumunu cebirsel ve analitik olarak incelerken karşılaştıkları zorlukları not etmişlerdir. Bu çalışmada planlanan eylem araştırmasında yapılan etkinliklerle bu yanılgıların giderilmesinde çözümler geliştirilmiş ve uygulanmıştır. Bu çalışmada Eylem araştırmasının, Planlama, uygulama, yansıtma ve değerlendirme aşamalarından oluşan uygulamanın araştırılması modeli kullanılmıştır (Çepni, 2010).

Bu çalışmaya Türkiye'deki bir devlet üniversitesinde Eğitim fakültesi ilköğretim Matematik Öğretmenliği programında öğrenim gören 42 üçüncü sınıf öğrencileri katılmıştır.

Çalışmada veriler yapılandırılmış sorular ve gözlem tekniğinden yararlanarak elde edilmiştir. Gözlemlerden ve etkinlikler boyunca öğrencilerden elde edilen veriler analiz edilirken, bilgiler gruplandırılarak yorumlanmıştır. Yapılandırılmış sorulardan elde edilen veriler nicel olarak analiz edilirken aynı zamanda nitel analizden yararlanılarak betimsel veri analizi yapılmıştır. Betimsel analizde öğrencilerin düşünce biçimlerini açık bir şekilde yansıtmak için öğrenci kağıtlarından alıntılarla örneklere yer verilmiş ve veriler orijinal haliyle aktarılmıştır.

Eylem araştırması süreci, problemin belirlenmesi ile başlayıp, plan yapma, planı uygulama ve uygulamanın değerlendirmesi aşamalarıyla devam etmiştir. Bu aşamalar aşağıda sırasıyla verilmiştir.

Problemin belirlenmesi: Her iki araştırmacıda farklı üniversitelerde yürüttükleri Uzay Analitik Geometri dersinde "Uzayda düzlemlerin konumları" konusunda sınıflarındaki önceki deneyimlerinden, gözlemlerinden yararlanarak “üç düzlemin birbirine gore konumlarının belirlenmesi” konusundaki problem durumunu belirlemişlerdir. Araştırmacılar uzayda düzlemler konusunun daha etkili bir öğretim yapabilmek ve plan hazırlamak için bir araya gelmişlerdir. “Uzayda düzlemlerin birbirine gore konumunu sadece cebirsel değil aynı zamanda görsel olarak ifade etme başarısını nasıl geliştirebiliriz?” sorusunun cevabı için plan yapma aşamasına geçilmiştir.

Plan yapma: Araştırmacılar üç aşamalı bir plan hazırlamışlardır. Birinci aşamada “ verilen üç düzlemin birbirine göre konumunun belirlenmesinde öğrenciler nasıl bir yol izlemektedirler?” sorusunun cevabı için farklı durumları içeren yapılandırılmış sorular hazırlamışlar ve bu yolla öğrencilerin ön bilgilerini belirlemek istemişlerdir. İkinci aşamada düzlemlerin birbirine göre konumlarının belirlenmesinde lineer denklem sistemlerinden yararlanma durumlarını belirlemek için farklı durumları içeren yapılandırılmış sorular hazırlamışlardır. Üçüncü aşamada ise düzlem denklemlerinin oluşturduğu lineer denklem sistemlerinin çözüm kümelerinin geometrik olarak yorumlanması ve düzlemlerin birbirine göre konumlarının görselleştirilebilmesi için Computer Algebra Systems (CAS) kullanarak etkinlikler planlamışlardır.

Planın Uygulanması: Oluşturulan plan çerçevesinde ders üç aşamadan oluşacak şekilde işlenmiştir. Araştırmacılardan biri sınıfın öğretmeni iken diğer araştırmacı uygulama sürecini, öğrencilerin verdiği cevapları ve öğretmenin sınıftaki rolünü video kamera ile kaydetmiştir. Uygulamalar interaktif tahtanın olduğu bir sınıfta gerçekleştirilmiştir. Uygulamadaki etkinlikler de interaktif tahtada öğrencilerle paylaşılmıştır. Ayrıca gözlemci araştırmacı tarafından planda ortaya çıkabilecek eksiklikler uygulamalar boyunca eleştirel gözle gözlemlenmiştir.

Uygulamanın Değerlendirilmesi: Araştırmanın bu aşamasında her iki araştırmacı bir araya gelerek alınan notlar, video kayıtları, yapılandırılmış sorulara verilen cevaplar incelenmiştir. Uygulama sürecinin değerlendirilmesi amacıyla yapılan bu incelemeler ile “hazırlanan planın öğrencilerin öğrenmesi üzerine etkisi nedir? Sorusunun cevabı aranmıştır. Değerlendirmeler sonucunda araştırmacılar etkinliklerin amacına ulaştığı konusunda fikir birliğine vardıkları için hazırlanan plan etkin bir plan olarak kabul edilmiş ve raporlaştırılmıştır.

Bu çalışmada planlanan araştırma süreci takip edilmiş ve uzayda düzlemlerin birbirine göre konumlarının belirlenmesinde yapılan hataların giderilmesi için düzenlenen etkinlikler çerçevesinde ders yürütülmüştür. Eylem planında belirlenen üç aşamalı uygulama adımlarından elde edilen bulgular sırasıyla aşağıda verilmiştir.

Uygulamanın 1. aşamasında öğretmen adaylarına üç düzlemin denklemleri verilerek düzlemlerin birbirine göre konumlarının belirlenmesi istenmiştir. Bu aşamada öğretmen adayları düzlemlerin birbirine göre konumunu ikişer ikişer düzlem denklemlerini inceleyerek tek yönlü belirlemeye çalışmıştır. Bu yöntem yanlış olmamakla birlikte eksik düşünmelerine neden olmuştur. Örneğin üç düzlemin bir tek noktada kesişmesi durumunu görmelerini engellemiştir. Uygulamanın ikinci aşamasında üç bilinmeyenli üç denklemden oluşan lineer denklem sistemlerinin çözümünü bulmaları ve bu çözümü geometrik olarak yorumlamaları istenmiştir. Bu aşamada ise öğretmen adaylarının bir kısmı denklem sistemini çözdüğü ancak geometrik olarak yorumlayamadığı gözlemlenmiştir. Bir kısmının ise denklem sisteminin katsayılar matrisinin determinantını hesaplayarak geometrik yorum yapmaya çalışmışlardır. Üçüncü aşamada ise üç düzlemin birbirine göre konumunun görselleştirmesi ve düzlem denklemlerini kullanarak bu düzlemlerin birbirlerine göre konumlarını gözlemleyebilmeleri için bilgisayar cebir sistemlerinden Maple programı kullanılmıştır. Bu aşamada öğretmen adayları düzlem denklemlerinin oluşturduğu denklem sistemlerinin çözüm kümesi ile Maple da elde edilen üç boyutlu görselleri ilişkilendirmişlerdir.

Yukarıda verilen uygulama aşamaları, öğrencilerin büyük çoğunluğunun daha önce yorumlayamadığı sistemin sonsuz çözümünün aslında tek parametreye bağlı bir çözüm olup bu parametrik çözümün düzlemlerin kesim doğrusunun parametrik denklemi olduğunu görmelerini sağladı. Düzlemlerin denklemlerinin bilgisayar programı aracılığıyla ekranda kesişim doğrusunu görmeleri, cebirsel sonucu geometrik görselle birleştirerek bilgiyi anlamlandırmalarını sağlamıştır. Ayrıca uygulama sayesinde düzlemlerin denklemlerinin oluşturduğu denklem sisteminin çözümü olmasa bile (çözüm kümesi boş küme) aslında verilen üç düzlemin sadece paralel olması gerekmediğini, düzlemlerin ikişer ikişer birer doğru boyunca kesişebileceğini görmüşlerdir. Bilgisayar programı CAS, öğrencilerin cebirsel çözüm ile geometrik temsili birleştirerek sonuca ulaşmalarına yardımcı olmuştur.