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A notion of robustness and stability of manifolds

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Abstract

Starting from the notion of thickness of Parks we define a notion of robustness for arbitrary subsets of $\mathbb{R}^k$ and we investigate its relationship with the notion of positive reach of Federer. We prove that if a set M is robust, then its boundary ∂M is of positive reach and conversely (under very mild restrictions) if ∂M is of positive reach, then M is robust. We then prove that a closed non-empty robust set in $\mathbb{R}^k$ (different from $\mathbb{R}^k$) is a codimension zero submanifold of class C^1 with boundary. As a partial converse we show that any compact codimension zero submanifold with boundary of class C^2 is robust. Using the notion of robustness we prove a kind of stability theorem for codimension zero compact submanifolds with boundary: two such submanifolds, whose boundaries are close enough (in the sense of Hausdorff distance), are diffeomorphic.

Keywords: Thickness; Positive reach; Stability of manifolds; Hausdorff distance

1. Introduction

Definition 1 *(ε-thick set).* A set $M \subset \mathbb{R}^k$ is called *ε-thick* (with $\varepsilon > 0$) if for every $x \in M$ there exists $y \in M$ such that $x \in \overline{B}(y, \varepsilon) \subset M$.

This notion is due to Parks [5] (see also [3]). ($B(y, \varepsilon)$ is the open ball with center y and radius ε, $\overline{B}(y, \varepsilon)$ is its closure.) We denote the set of ε-thick subsets of $\mathbb{R}^k$ by $\mathcal{T}_\varepsilon$.

Definition 2 *(ε-robust set).* We call a set $M \subset \mathbb{R}^k$ *ε-robust* (with $\varepsilon > 0$), if M and $M' = \mathbb{R}^k \backslash M$ are both ε-thick.

We denote the set of ε-robust subsets of $\mathbb{R}^k$ by $\mathcal{R}_\varepsilon$:

$$\mathcal{R}_\varepsilon = \{M \mid M \in \mathcal{T}_\varepsilon \text{ and } M' \in \mathcal{T}_\varepsilon\}.$$

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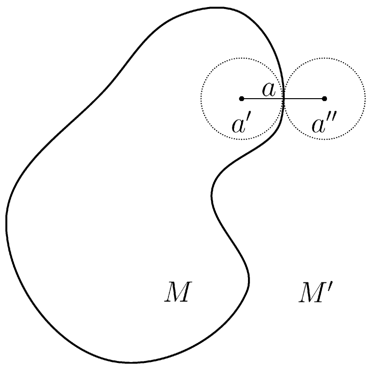

Fig. 1. The associated segment $[a'a'']$ to the point $a \in \partial M$.

We call a set $M \subset \mathbb{R}^k$ *robust*, if it is ε-robust for some $\varepsilon > 0$ and define its *robustness* as

$$\mathrm{robustness}(M) = \sup\{\varepsilon > 0 \mid M \in \mathcal{R}_\varepsilon\}.$$

Definition 3 *(ε-reach set).* Let $\emptyset \neq S \subset \mathbb{R}^k$ and $\varepsilon > 0$. S is said to be of *ε-reach*, if for all $x \in \mathbb{R}^k$ with $\mathrm{dist}(x, S) < \varepsilon$ (where $\mathrm{dist}(x, S) = \inf\{|x - s|\colon\ s \in S\}$), there exists a unique point $a \in S$ such that $\mathrm{dist}(x, S) = |x - a|$.

This notion is due to Federer [1]. See also [4].

Define the ε-neighborhood of S as

$$N(S, \varepsilon) = \bigl\{x \in \mathbb{R}^k \bigm| \mathrm{dist}(x, S) < \varepsilon\bigr\}.$$

For an ε-reach set S, we have the projection map (called the *nearest point map*)

$$\pi : N(S, \varepsilon) \to S$$

defined with the notations above as $\pi(x) = a$ for all $x \in N(S, \varepsilon)$. Note that for all points y on the line segment between x and a, $\pi(y) = \pi(x) = a$.

Lemma 4. *Let $M \in \mathcal{R}_\varepsilon$ and $a \in \partial M$. Then there exist unique points $a' \in M$ and $a'' \in M'$ with $|a' - a| = |a'' - a| = \varepsilon$ such that $B(a', \varepsilon) \subset M$ and $B(a'', \varepsilon) \subset M'$.*

Proof. Let $a \in \partial M$ belong to M. (The case $a \in M'$ can be handled similarly.) Then $M \in \mathcal{T}_\varepsilon$ implies the existence of a point $a' \in M$ with $a \in \overline{B}(a', \varepsilon) \subset M$. a must lie on the boundary of the disk $\overline{B}(a', \varepsilon)$, because otherwise a would be an interior point of M. We now choose a point a'' as the point on the line $a'a$ such that a is the middle point of the closed segment $[a'a'']$ (see Fig. 1).

In order to prove $B(a'', \varepsilon) \subset M'$, consider a sequence (a_n) of points in M' converging to a and for each n let c_n be a point such that $a_n \in \overline{B}(c_n, \varepsilon) \subset M'$. By restricting to a subsequence if necessary we can assume that (c_n) converges to a point c. Since $\mathrm{dist}(c_n, M) \geqslant \varepsilon$ we conclude that $\mathrm{dist}(c, M) \geqslant \varepsilon$, so $B(c, \varepsilon) \subset M'$. As $|a_n - c_n| \leqslant \varepsilon$ we conclude that $|a - c| \leqslant \varepsilon$, but since $a \in M$, we get $|a - c| = \varepsilon$, and so $|a' - c| \leqslant 2\varepsilon$. Since $\overline{B}(a', \varepsilon) \cap B(c, \varepsilon) = \emptyset$, then $|a' - c| \geqslant 2\varepsilon$. Thus the distance the $|a' - c| = 2\varepsilon$ and $c = a''$. We have thus proved that $B(a'', \varepsilon) \subset M'$.

In order to prove the uniqueness, assume the point $a \in \partial M$ admits two pairs of points (a', a'') and $(\bar{a}', \bar{a}'')$ with the required properties. Since $B(a', \varepsilon) \cap B(a'', \varepsilon) = \emptyset$, $|a' - a''| \geqslant 2\varepsilon$ and since $|a' - a| = | a'' - a| = \varepsilon$, then $|a' - a'' | \leqslant 2\varepsilon$. We conclude that $|a' - a''| = 2\varepsilon$ and that a is the midpoint of $[a'a'']$. Similarly a is the midpoint of $[\bar{a}'\bar{a}'']$.

Obviously the pair of points $(a', \bar{a}'')$ also satisfies the required properties in the statement of the lemma, then as above we conclude that a is the midpoint of $[a'\bar{a}'']$ as well. Then $\bar{a}'' = a''$ and $\bar{a}' = a'$. $\square$

Lemma 4 enables us to associate a closed segment $[a'a'']$ to every point $a \in \partial M$ such that a is the middle point of this segment. Points of $[a'a)$ are interior points of M and points of $(aa'']$ are interior points of M'.

Remark 5. With the notations of Lemma 4, for any point $c \in [a'a'']$, $\mathrm{dist}(c, \partial M) = |c - a|$ holds.

Proof. Let us first see $\mathrm{dist}(a', \partial M) = \varepsilon$. Assume there exists $b \in \partial M$ with $|a' - b| < \varepsilon$. Then, because of $b \in B(a', \varepsilon) \subset M$, b would be an interior point of M, contradicting $b \in \partial M$. Similarly, $\mathrm{dist}(a'', \partial M) = \varepsilon$.

Now let $c \in [a'a)$, and assume $|c-b| < |c-a|$ for some $b \in \partial M$. Then we would get $|a'-b| \leqslant |a'-c| + |c-b| < |a'-c| + |c-a| = \varepsilon$, thus $\mathrm{dist}(a', \partial M) < \varepsilon$ contradicting $\mathrm{dist}(a', \partial M) = \varepsilon$. □

We will denote the set $[a'a''] - \{a', a''\}$ by $(a'a'')$ and call it the *open segment associated to* $a \in \partial M$. We will call $(a'a) \subset (a'a'')$ the *inner part* of $(a'a'')$ and $(aa'') \subset (a'a'')$ the *outer part*.

We will now show that open segments associated to different points are disjoint:

Lemma 6. *Let $M \in \mathcal{R}_\varepsilon$, $a, b \in \partial M$, $a \neq b$ and let $(a'a'')$, $(b'b'')$ be the open segments associated to a and b. Then $(a'a'') \cap (b'b'') = \emptyset$.*

Proof. We consider two cases:

(i) Let $(a'a'')$ and $(b'b'')$ intersect at a single point c. Then c must be different from both a and b. If we had $c = a$, then $\mathrm{dist}(c, \partial M) = 0$. By Remark 5, $|c - b| = \mathrm{dist}(c, \partial M)$=0, so $c = b$, contradicting $a \neq b$.

c must lie either on the outer parts of the segments or on the inner parts. Otherwise, c would be at the same time an interior point of M and M'.

Assume, for example, $c \in (aa'') \cap (bb'')$. Then $|b - a''| < |b - c| + |c - a''|$ since b, c and a'' are not collinear. By Remark 5, we have $|b - c| = \mathrm{dist}(c, \partial M) = |a - c|$.

Hence, $|b - a''| < |a - c| + |c - a''| = |a - a''| = \varepsilon$. But this contradicts $\mathrm{dist}(a'', \partial M) = \varepsilon$.

(ii) If $(a'a'')$ and $(b'b'')$ intersect along a common subsegment, then either b' or b'' must belong to $(a'a'')$. But that would give a distance to a less than ε, contradicting the fact that both b' and b'' are at ε distance to ∂M. □

We now show that the union of open segments associated to the points of ∂M is exactly the ε-neighborhood of ∂M.

Lemma 7. *Let $M \in \mathcal{R}_\varepsilon$ and $(a'a'')$ the open segment associated to $a \in \partial M$. Then $\bigcup_{a \in \partial M} (a'a'') = N(\partial M, \varepsilon)$.*

Proof. By Remark 5, for any $c \in (a'a'')$ we have $\mathrm{dist}(c, \partial M) = |c - a| < \varepsilon$. This shows $c \in N(\partial M, \varepsilon)$, i.e. $\bigcup_{a \in \partial M} (a'a'') \subset N(\partial M, \varepsilon)$.

To see the other inclusion, let $c \in N(\partial M, \varepsilon)$. Since ∂M is closed, there exists a point $a \in \partial M$ realizing the $\mathrm{dist}(c, \partial M)$. It can be $c \in M$ or $c \in M'$. Assume $c \in M'$ (the other case being similar). We want to show $c \in [aa'')$.

Since $B(a', \varepsilon) \subset M$ we have

$$|c - a'| \geqslant \mathrm{dist}\big(c, B(a', \varepsilon)\big) + \varepsilon \geqslant \mathrm{dist}(c, M) + \varepsilon = \mathrm{dist}(c, \partial M) + \varepsilon = |c - a| + \varepsilon.$$

On the other hand

$$|c - a'| \leqslant |c - a| + |a - a'| = |c - a| + \varepsilon.$$

Thus $|c - a'| = |c - a| + |a - a'|$ and the three points c, a, a' are collinear. Since $c \in M'$, then $c \in [aa'')$. □

2. Relation with the concept of positive reach

We now investigate the relationship between robustness and positive reach.

Theorem 8. *If $M \subset \mathbb{R}^k$ is ε-robust, then ∂M is of ε-reach.*

Proof. ($\Rightarrow$) Let $M \in \mathcal{R}_\varepsilon$, $c \in N(\partial M, \varepsilon)$ and $a \in \partial M$ and $b \in \partial M$ be two points realizing $\mathrm{dist}(c, \partial M)$. Then, by the proof of Lemma 7 $c \in (a'a'')$ and $c \in (b'b'')$. But by Lemma 6 $(a'a'')$ and $(b'b'')$ are disjoint if $a \neq b$. So we get $a = b$, showing that ∂M is of ε-reach. □

Theorem 9. *Let $M \subset \mathbb{R}^k$ with $\overline{\mathring{M}} = M$ (i.e. M is the closure of its interior points). If ∂M is of ε-reach, then M is δ-robust for all $0 < \delta < \varepsilon$.*

Proof. We have to show that M and $M' = \mathbb{R}^k \backslash M$ are both δ-thick for $0 < \delta < \varepsilon$. We show below that M is δ-thick. (Similarly it can be shown that M' is δ-thick.)

If $x \in M \backslash N(\partial M, \varepsilon)$, then $\overline{B}(x, \delta) \subset M$.

Now let $x \in \mathring{M} \cap N(\partial M, \varepsilon)$. As ∂M is of ε-reach, there exists a unique $\xi \in \partial M$ with $|x - \xi| = \operatorname{dist}(x, \partial M)$.

We first recall that for any other point y on the segment $(x\xi)$ the nearest point on ∂M is again ξ. If, on the contrary, $\eta \in \partial M$ were the nearest point to y, then the inequalities

$$|x - \eta| \leqslant |x - y| + |y - \eta| < |x - y| + |y - \xi| = |x - \xi|$$

would give a contradiction. By this argument, for any point $z \in \mathring{M} \cap N(\partial M, \varepsilon)$ collinear with x and ξ, the associated nearest point on ∂M must be again ξ. We next show that we can define the point $x' \in M$ with the properties $x \in \overline{B}(x', \delta) \subset M$.

Let $x' \in \mathring{M} \cap N(\partial M, \varepsilon)$ be the point collinear with x and ξ and with distance $\delta + (1 - \frac{\delta}{\varepsilon})d$ to ξ where $d = |x - \xi|$. It can be easily verified that $x \in \overline{B}(x', \delta) \subset M$.

As the last case, assume $x \in \partial M$. Because of $\overline{\mathring{M}} = M$, there exists a sequence $\{x_n\}$ of interior points of M converging to x. For every x_n, there exists according to the preceding cases $x'_n \in M$ with $x_n \in \overline{B}(x'_n, \delta) \subset M$.

Choose a converging subsequence of $\{x'_n\}$, say $\{y_n\}$, with limit $y \in \overline{\mathring{M}}$. Then we show that $x \in \overline{B}(y, \delta) \subset M$.

Assume $x \notin \overline{B}(y, \delta)$. Then we would have $B(x, \rho) \cap \overline{B}(y, \delta) = \emptyset$ for some $\rho > 0$. Then a closed disk with radius δ, whose center is closer to y than $|x - y| - \rho - \delta$, would not intersect $B(x, \rho)$ also. This means that, for y_n close enough to y, the corresponding x_n would lie outside of $B(x, \rho)$, contradicting the convergence $x_n \to x$.

$\overline{B}(y, \delta) \subset M$ can be shown similarly: If $\overline{B}(y, \delta)$ contains a point of M', then for y_m close enough to y, $\overline{B}(y, \delta)$ would also contain a point of M'. □

3. Robustness of codimension zero submanifolds

In this section we give some relations between robustness and smoothness in $\mathbb{R}^k$. We use the notion of submanifold with boundary in the sense of Hirsch [2, p. 30]. Codimension of a submanifold is the difference between the dimension of the ambient manifold and the dimension of the submanifold. Thus, the term "codimension zero" means that the submanifold has top dimension (i.e. its dimension equals the dimension of the ambient space).

Theorem 10. *Let $M \subset \mathbb{R}^k$ be a compact codimension zero submanifold with boundary of differentiability class C^s where $s \geqslant 2$. Then M is robust. Moreover, the map*

$$\Phi : \partial M \times (-r, r) \to N(\partial M, r)$$

given by $\Phi(a, x) = a + \frac{x}{r}\overrightarrow{aa''}$ is a C^{s-1} diffeomorphism, where $r = \operatorname{robustness}(M)$.

Proof. By [4, Theorem 4.4.10], the boundary ∂M has positive reach. The conditions of Theorem 9 are satisfied, hence $M \in \mathcal{R}_\varepsilon$ for some $\varepsilon > 0$.

Consider the Weingarten map at $a \in \partial M$ sending $w \in T_a(\partial M)$ to $-N'_w(a)$, where $T_a(\partial M)$ denotes the tangent space of ∂M at a and $N'_w(a)$ denotes the derivative of the outer normal vector field of ∂M at a in the direction of w. Here obviously the outer (unit) normal vector $N(a)$ at a equals $\frac{1}{r}\overrightarrow{aa''}$.

Fix a point $a \in \partial M$ and let $(w_1, w_2, \dots, w_{k-1})$ (respectively $(\kappa_1, \kappa_2, \dots, \kappa_{k-1})$), be the principal directions (respectively principal curvatures) at a.

Choose a local coordinate system $(u_1, u_2, \dots, u_{k-1})$ on ∂M near a such that

$$\frac{\partial}{\partial u_i}(a) = w_i, \quad i = 1, \dots, k-1.$$

Then the Weingarten map at a

$$w \to -N'_w(a)$$

is diagonal with respect to the basis $(w_1, w_2, \dots, w_{k-1})$ in $T_a(\partial M)$, i.e.

$$-N'_{w_i}(a) = \kappa_i w_i, \quad i = 1, \dots, k-1.$$

Since robustness$(M) = r > 0$, we conclude that for all $i = 1, \dots, k-1$

$$0 \leqslant |\kappa_i| \leqslant \frac{1}{r}$$

by considering the normal section of ∂M at a along w_i.

Since ∂M is of r-reach the map Φ is injective and by Lemma 7 (applied for all $0 < \varepsilon < r$), Φ is also onto. Since the normal vector $N(a)$ is of class C^{s-1}, it is sufficient to prove that the map Φ is regular on $\partial M \times (-r, r)$.

With the notations above

$$\frac{\partial \Phi}{\partial u_i}(a, x) = \frac{\partial}{\partial u_i}(a) - x\bigl(-N'_{w_i}(a)\bigr) = w_i - x\kappa_i w_i = (1 - x\kappa_i)w_i \quad \text{for all } i = 1, \dots, k-1,$$

and

$$\frac{\partial \Phi}{\partial x}(a, x) = \frac{1}{r}\overrightarrow{aa''} = N(a).$$

Since $1 - x\kappa_i \neq 0$ for all $x \in (-r, r)$, we conclude that Φ is regular at (a, x). $\square$

Remark 11. If the differentiability class of M in Theorem 10 is less smooth than C^2, then the conclusion need not hold, i.e. M might not be robust. A counter-example can be manufactured by using Example 4.4.12 in [4].

Remark 12. Under the conditions of Theorem 10, we conclude also that the nearest point map $\pi : N(\partial M, r) \to \partial M$ is of class C^{s-1} on the whole tubular neighborhood $N(\partial M, r)$, since $\pi = \mathrm{pr}_1 \circ \Phi^{-1}$, where pr_1 is the projection of $\partial M \times (-r, r)$ on its first factor.

Remark 13. From the proof of Theorem 10, we obtain the estimate

$$\text{robustness}(M) \leqslant \frac{1}{\max\{|\kappa_w(a)|:\ w \in T_a(\partial M),\ a \in \partial M,\ w \neq 0\}}$$

where $\kappa_w(a)$ is the normal curvature of the hypersurface ∂M at a in the direction of w.

Theorem 14. *Let $M \subset \mathbb{R}^k$ be closed, non-empty and $M \neq \mathbb{R}^k$. If M is robust, then M is a codimension zero submanifold with boundary of differentiability class C^1.*

Proof. As M is ε-robust for some $\varepsilon > 0$, we can apply Lemma 7:

$$N(\partial M, \varepsilon) = \bigcup_{a \in \partial M} (a'a'').$$

To make the dependence of a'' on a one-to-one, we use a smaller δ-neighborhood with $\delta < \varepsilon$ and denote the segment again by $(a'a'')$:

$$N(\partial M, \delta) = \bigcup_{a \in \partial M} (a'a'').$$

We define the following function (called the *signed-distance* to ∂M):

$$f_{\partial M} : \mathbb{R}^k \to \mathbb{R},$$

$$f_{\partial M}(x) = \begin{cases} \operatorname{dist}(x, \partial M) & \text{if } x \in M', \\ -\operatorname{dist}(x, \partial M) & \text{if } x \in M. \end{cases}$$

Since any $x \in N(\partial M, \delta)$ belongs to a unique segment $(a'a'')$ associated to $a \in \partial M$, the restriction of $f_{\partial M}$ to $N(\partial M, \delta)$, say f, is given by the formula:

$$f(x) = \begin{cases} |x - a| & \text{for } x \in [aa''), \\ -|x - a| & \text{for } x \in (a'a]. \end{cases}$$

We will show that f is of class C^1 with $|\nabla f| = 1$ ($\nabla = \text{grad}$) on the whole domain. As $\partial M = f^{-1}(0)$, this will yield the theorem.

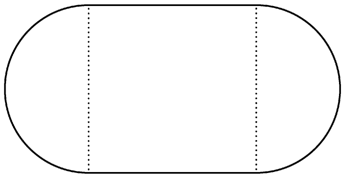

Fig. 2. Stadium.

To this end, after fixing a segment $(a'a'')$, we consider two auxiliary functions f_1 and f_2 as signed-distance function to the δ-spheres around a' respectively a'', i.e. $f_1(x) = |x - a'| - \delta$ respectively $f_2(x) = |x - a''| - \delta$ for $x \in N(\partial M, \delta)$.

It is easily seen that the inequalities $-f_2 \leqslant f \leqslant f_1$ hold. Likewise, it can be computed that $(\nabla f_1)(x) = \frac{a''-a}{\delta}$ and $(\nabla f_2)(x) = -\frac{a''-a}{\delta}$ for $x \in (a'a'')$.

As $-f_2 \leqslant f \leqslant f_1$, $-f_2(x) = f(x) = f_1(x)$ for $x \in (a'a'')$ and $(\nabla(-f_2))(x) = (\nabla f_1)(x)$ for $x \in (a'a'')$, we find that $(\nabla f)(x) = \frac{a''-a}{\delta}$ and $|(\nabla f)(x)| = 1$.

Continuity of $\frac{a''-a}{\delta}$ as it depends on $x \in N(\partial M, \delta)$ can be shown using [4, Lemma 4.4.3], and locally inverting the function $a \mapsto a''$. $\square$

Remark 15. The differentiability class in the conclusion of the Theorem 14 need not be higher than C^1. As an example consider $M \subset \mathbb{R}^2$ as indicated in Fig. 2 (as a union of a square and two half-disks).

4. A stability theorem of submanifolds

We will now prove a kind of stability theorem for codimension zero submanifolds with boundary in $\mathbb{R}^k$. By Theorem 10, a compact codimension zero submanifold with boundary in $\mathbb{R}^k$ of class C^2 has positive robustness.

Theorem 16. *Let M and N be compact codimension zero submanifolds with boundary in $\mathbb{R}^k$ of class C^s (with $s \geqslant 2$) and let*

$$r = \min\{\text{robustness}(M), \text{robustness}(N)\}.$$

If $d_H(\partial M, \partial N) < \frac{r}{4}$, where d_H denotes the Hausdorff distance, then ∂M and ∂N are diffeomorphic of class C^s. Furthermore, M and N are diffeomorphic of class C^s.

Proof. Using Lemma 4 and Remark 5, one can see that for any $a \in \partial M$ there exist unique points $a' \in M$ and $a'' \in M'$ such that

(i) $\overline{B}(a', r) \subset M$ (as M is closed).
(ii) $B(a'', r) \subset M'$.
(iii) $\overline{B}(a', r) \cap \overline{B}(a'', r) = \{a\}$.

We consider the closed balls $\alpha' = \overline{B}(a', \frac{3}{4}r)$ and $\alpha'' = \overline{B}(a'', \frac{3}{4}r)$. We have $d_H(\alpha', \partial M) \geqslant \frac{r}{4}$ and $d_H(\alpha'', \partial M) \geqslant \frac{r}{4}$. The hypothesis $d_H(\partial M, \partial N) < \frac{r}{4}$ implies $\partial N \cap (\alpha' \cup \alpha'') = \emptyset$.

Claim 1. $(a'a'') \cap \partial N \neq \emptyset$.

Assume to the contrary that $(a'a'') \cap \partial N = \emptyset$. In that case, the set $\alpha' \cup \alpha'' \cup (a'a'')$ does not intersect ∂N.

As $\alpha' \cup \alpha'' \cup (a'a'')$ is connected, it is either contained in N or in N'. Consider the case $\alpha' \cup \alpha'' \cup (a'a'') \subset N$. (The other case can be handled similarly.)

As $a \in \partial M$ and $d_H(\partial M, \partial N) < \frac{r}{4}$, there exists $b \in \partial N$ such that $a \in B(b, \frac{r}{4})$. For this $b \in \partial N$, there exist unique b' and b'' such that

(i) $\overline{B}(b', r) \subset N$, (ii) $B(b'', r) \subset N'$, (iii) $\overline{B}(b', r) \cap \overline{B}(b'', r) = \{b\}$.

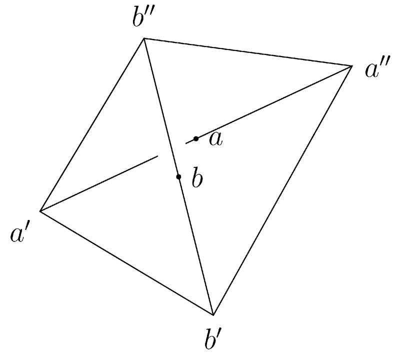

Fig. 3. The pair of not necessarily intersecting segments $[a'a'']$ and $[b'b'']$ in $\mathbb{R}^k$.

We define $\beta' = \overline{B}(b', \frac{3}{4}r)$ and $\beta'' = \overline{B}(b'', \frac{3}{4}r)$. As above, we have $\partial M \cap (\beta' \cup \beta'') = \emptyset$.

$$\alpha' \cup \alpha'' \cup (a'a'') \subset N \quad \text{and} \quad B(b'', r) \subset N' \quad \text{imply} \quad \big(\alpha' \cup \alpha'' \cup (a'a'')\big) \cap B(b'', r) = \emptyset.$$

From $\operatorname{dist}(a', \partial N) \geqslant \frac{3}{4}r$, $\operatorname{dist}(a'', \partial N) \geqslant \frac{3}{4}r$ and $\operatorname{dist}(b'', \partial N) = r$ we get $|a' - b''| \geqslant \frac{7}{4}r$ and $|a'' - b''| \geqslant \frac{7}{4}r$. Now consider the pair of segments in $\mathbb{R}^k$, $[a'a'']$ and $[b'b'']$ (see Fig. 3).

We can then write the "quadrilateral" identity:

$$|a' - b'|^2 + |b' - a''|^2 + |a'' - b''|^2 + |b'' - a'|^2 = |a' - a''|^2 + |b' - b''|^2 + 4|a - b|^2.$$

$$\begin{aligned} |a' - b'|^2 + |a'' - b'|^2 &= |a' - a''|^2 + |b' - b''|^2 + 4|a - b|^2 - |a' - b''|^2 - |a'' - b''|^2 \\ &\leqslant (2r)^2 + (2r)^2 + 4\left(\frac{r}{4}\right)^2 - \left(\frac{7}{4}r\right)^2 - \left(\frac{7}{4}r\right)^2 \\ &\leqslant \frac{34}{16}r^2 < \left(\frac{7}{4}r\right)^2. \end{aligned}$$

This means that $|a' - b'| < \frac{7}{4}r$ and $|a'' - b'| < \frac{7}{4}r$, implying $\overline{B}(b', \frac{3}{4}r) \cap B(a', r) \neq \emptyset$ and $\overline{B}(b', \frac{3}{4}r) \cap B(a'', r) \neq \emptyset$, i.e. $\beta' \cap B(a', r) \neq \emptyset$ and $\beta' \cap B(a'', r) \neq \emptyset$. The connected set β' must than intersect ∂M, because it intersects $B(a', r) \subset M$ and $B(a'', r) \subset M'$. But this is a contradiction, as we know $\partial M \cap (\beta' \cup \beta'') = \emptyset$. This excludes the possibility $\alpha' \cup \alpha'' \cup (a'a'') \subset N$. Consequently, $(a'a'')$ must intersect ∂N, so Claim 1 is proved.

Let us choose a point in $(a'a'') \cap \partial N$ and denote it by n. Since $\operatorname{dist}(n, \partial M) = |n - a|$ and $d_H(\partial M, \partial N) < \frac{r}{4}$, we have $|n - a| < \frac{r}{4}$.

Claim 2. $(a'a'') \cap \partial N = \{n\}$.

Consider the quadrilateral $a'n'a''n''$, where $[n'n'']$ is the associated segment to $n \in \partial N$ with $|n - n'| = |n - n''| = r$ (see Fig. 4). By the argument used in the end of the proof of Claim 1, consecutive edges of the quadrilateral $a'n'a''n''$ cannot be both less than $\frac{7}{4}r$. On the other hand, by the quadrilateral identity, we get

$$\begin{aligned} |a' - n'|^2 + |n' - a''|^2 + |a'' - n''|^2 + |n'' - a'|^2 &= |a' - a''|^2 + |n' - n''|^2 + 4|a - n|^2 \\ &\leqslant (2r)^2 + (2r)^2 + 4\left(\frac{r}{4}\right)^2 \\ &\leqslant \frac{132}{16}r^2 < 3\left(\frac{7}{4}r\right)^2 \end{aligned}$$

which means that at most two of the sides of the quadrilateral can be greater than or equal to $\frac{7}{4}r$. In conclusion, exactly two sides of $a'n'a''n''$ must be at least of length $\frac{7}{4}r$ and these two sides must be opposite sides of the quadrilateral, say for instance, $[a'n'']$ and $[a''n']$.

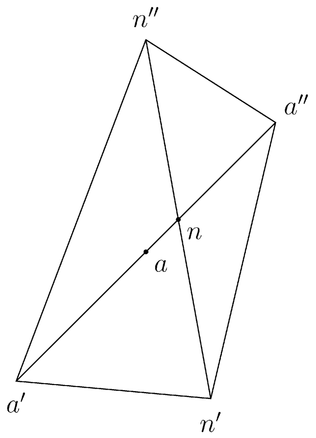

Fig. 4. The pair of intersecting segments $[a'a'']$ and $[n'n'']$.

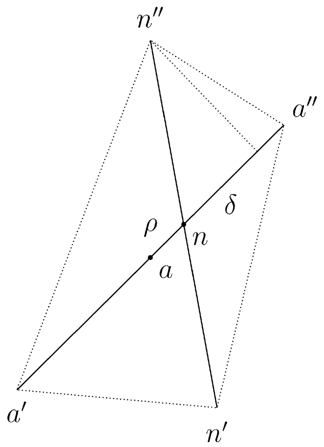

Fig. 5. Projection of the segment $[nn'']$ onto the line $a'a''$.

Let δ be the length of the projection of $[nn'']$ on the line $a'a''$ (see Fig. 5). We will show $\delta \geqslant \frac{r}{2}$. We abbreviate $|a-n|$ by ρ. We can write

$$2\left(\frac{7}{4}r\right)^2 \leqslant |a'-n''|^2 + |a''-n'|^2 = (r+\rho)^2 + r^2 + 2(r+\rho)\delta + (r-\rho)^2 + r^2 + 2(r-\rho)\delta$$
$$= 4r^2 + 2\rho^2 + 4r\delta,$$

$$2r^2 + \rho^2 + 2r\delta \geqslant \frac{49}{16}r^2,$$
$$2r\delta \geqslant \left(\frac{49}{16} - 2 - \frac{1}{16}\right)r^2 = r^2 \quad \text{since } \rho = |a-n| < \frac{r}{4},$$

which gives $\delta \geqslant \frac{r}{2}$.

Now we can see that n is the unique point in $(a'a'') \cap \partial N$. Indeed, if there was another such point $\bar{n}$, we would have

$$|n-\bar{n}| \leqslant |n-a| + |a-\bar{n}| < \frac{r}{2}.$$

Since $\bar{n} \in \partial N$, thus $\bar{n} \notin B(n', r) \cup B(n'', r)$ we must have $|n-\bar{n}| \geqslant 2\delta \geqslant r$ which contradicts $|n-\bar{n}| < \frac{r}{2}$, proving Claim 2.

From Claim 2 it follows that we have a well-defined map:

$$\varphi : \partial M \to \partial N,$$
$$\varphi(a) = n, \quad \text{with } n \in (a'a'') \cap \partial N.$$

Since the associated open segments to different points of ∂M are disjoint, the map φ is injective.

On the other hand, the nearest point map $\pi : N(\partial M, r) \to \partial M$, gives us a map

$$\psi = \pi|_{\partial N} : \partial N \to \partial M.$$

φ and ψ are inverse to each other. As ψ is of class C^{s-1}, it is enough to prove that φ is also of class C^{s-1}. This would mean that the C^s-submanifolds ∂M and ∂N are C^{s-1}-diffeomorphic and consequently automatically C^s-diffeomorphic (see [2, Theorem 2.2.10(b)]).

Claim 3. φ is of class C^{s-1}.

Consider a local C^s system of coordinates $(u_1, \dots, u_{k-1})$ on ∂M around $a \in \partial M$. Then, we can find a local C^{s-1} system of coordinates $(u_1, \dots, u_{k-1}, u_k = f)$ on an open set $U \subset \mathbb{R}^k$ around the point $a \in \partial M$, where f is the signed-distance function to ∂M. Then there exists a C^{s-1} function g on U, such that,

(i) $U \cap \partial N = \{x \in U \mid g(x) = 0\}$.
(ii) $\nabla g(x) \neq 0$ for all $x \in U \cap \partial N$.

In order to prove C^{s-1}-smoothness of φ, it is enough to show $\frac{\partial g}{\partial u_k}(n) \neq 0$ for all $n \in U \cap \partial N$, since

$$\text{for all } m \in \partial M, \quad u_i\big(\varphi(m)\big) = u_i(m), \quad i = 1, \dots, k-1.$$

Indeed, by Lemma 4.4.4 in [4], $\nabla u_k(n) = \frac{1}{r}\overrightarrow{aa''}$ and $\nabla g(n) = |\nabla g(n)|\frac{1}{r}\overrightarrow{nn''}$. (As $\nabla g(n)$ is an outer normal vector for N at $n \in \partial N$.)

Since $|\langle \overrightarrow{nn''}, \frac{1}{r}\overrightarrow{aa''}\rangle| = \delta \geqslant \frac{r}{2} > 0$, the vectors $\overrightarrow{nn''}$ and $\overrightarrow{aa''}$ are not perpendicular. Thus

$$\begin{aligned}
\left|\frac{\partial g}{\partial u_k}(n)\right| &= \left|\left\langle \big|\nabla g(n)\big|\frac{1}{r}\overrightarrow{nn''}, \frac{1}{r}\overrightarrow{aa''}\right\rangle\right| \\
&= \frac{|\nabla g(n)|}{r}\left|\left\langle \overrightarrow{nn''}, \frac{1}{r}\overrightarrow{aa''}\right\rangle\right| \\
&= \frac{|\nabla g(n)|}{r}\delta > 0
\end{aligned}$$

for all $n \in \partial N \cap U$, which finishes the proof of Claim 3.

To prove the last statement of the theorem (that M and N are diffeomorphic of class C^s), we will use the following technical fact, whose proof can be given by standard arguments:

There exists a C^∞ function

$$h : [0,1] \times \mathbb{R} \times \left(-\frac{r}{4}, \frac{r}{4}\right) \to \mathbb{R}$$

with the following properties:

(i) $h(0, x, y) = x$ for all $x \in \mathbb{R}$, $y \in (-\frac{r}{4}, \frac{r}{4})$;
(ii) $h(1, 0, y) = y$ for all $y \in (-\frac{r}{4}, \frac{r}{4})$;
(iii) $h(t, x, y) = x$ for all $x \in \mathbb{R} \setminus (-\frac{3}{4}r, \frac{3}{4}r)$, $y \in (-\frac{r}{4}, \frac{r}{4})$, $t \in [0,1]$;
(iv) $h(t, \cdot, y) : \mathbb{R} \to \mathbb{R}$ is an increasing C^∞-diffeomorphism for all $t \in [0,1]$ and $y \in (-\frac{r}{4}, \frac{r}{4})$.

Using the function $\varphi : \partial M \to \partial N$ defined and investigated above, we define the function $y : \partial M \to (-r, r)$ by

$$y(a) = \begin{cases} |a - \varphi(a)| & \text{if } \varphi(a) \in M', \\ -|a - \varphi(a)| & \text{if } \varphi(a) \in M. \end{cases}$$

As φ is C^{s-1}, y is also a C^{s-1}-function because $y(a) = \mathrm{dist}(\varphi(a), \partial M) = f(\varphi(a))$ and f is also of class C^{s-1}.

Let $\Phi : \partial M \times (-r, r) \to N(\partial M, r)$ be the C^{s-1}-diffeomorphism in Theorem 10

$$\Phi(a, x) = a + \frac{x}{r}\overrightarrow{aa''} \quad \bigl(a \in \partial M,\ x \in (-r, r)\bigr).$$

We can define the isotopy $F_t : \mathbb{R}^k \to \mathbb{R}^k$ using the function h as follows:

$$F_t(p) = p, \quad \text{for all } t \in [0,1] \text{ and } p \in \mathbb{R}^k \setminus N\left(\partial M, \frac{3}{4}r\right),$$

and

$$F_t\bigl(\Phi(a, x)\bigr) = \Phi\bigl(a, h\bigl(t, x, y(a)\bigr)\bigr) \quad \text{for all } t \in [0,1] \quad \text{and} \quad p = \Phi(a, x) \in N(\partial M, r).$$

Thus:

(i) $F_0(p) = p$, for all $p \in \mathbb{R}^k$;
(ii) $F_1(a) = \varphi(a)$, for all $a \in \partial M$ (giving $F_1(\partial M) = \partial N$);
(iii) For all $a \in \partial M$, the function $x \in (-r, r) \mapsto h(1, x, y(a))$ is increasing, giving $F_1(M) = N$;
(iv) F_t is a C^1-diffeomorphism for all $t \in [0,1]$.

M and N, being C^{s-1}-diffeomorphic C^s-manifolds, are C^s-diffeomorphic by [2], Theorem 2.2.10(b). $\square$

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