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8th Grade Students' Construction Processes of the Concept of Slope *

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Abstract

Slope is a fundamental concept for numerous higher mathematical concepts, especially for derivative. Moreover, this concept is a cornerstone in many different areas such as art, architecture, engineering, and physics. Prior to learning the concept of slope formally, students have informal knowledge because of its use in daily life. However, students learn the concept formally in eighth grade for the first time. The purpose of this study was to investigate eighth grade students' construction processes of the concept of slope. This study is a part of a dissertation research, which focused on designing a teaching process of slope and investigating students' mathematizing and construction process of the concept. The participants of this study were five eighth grade students. These students were chosen based on the results of an open-ended test related to the concepts prerequisite for the concept of slope. The study was designed as a qualitative study. Data of this study were collected by 15 clinical interviews and participants were interviewed three times. Data were analyzed by thematic analysis method and participants' stages of comprehension the concept of slope were interpreted in the context of APOS theory.

Keywords

Slope
APOS
Conceptualization
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Stages of comprehension

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Introduction

The concept of slope is encountered in many different situations such as ramps, piedmonts, roofs and so on in our everyday life from the beginning of the school. The formal construction process of the concept of slope starts at middle grades (Hoffman, 2015). The slope that arises from the need to measure the perpendicularity of the linear models encountered in real life (Sandoval, 2013) is the ratio of the distance taken in the horizontal direction to the distance taken in the vertical direction on the linear visual (linear models such as piedmont, ramp and road) (Lobato & Thanheiser, 2002). Construction of the slope as a rate starting from the middle school years and in later grades the continuation of the formation process by making meaningful transitions between the different representations seems important in forming many related concepts, especially the derivative (Clement, 1985; Stump, 1999; Tabaghi, Mamolo, & Sinclair, 2009; Cheng, 2010; Moore-Russo, Conner, & Rugg, 2011). Nagle and Moore-Russo (2014) states that students' understanding slope as a constant rate of change in middle years and as an average rate of change in high school years allows them to get ready for the idea of instantaneous rate of change and the concept of derivative. Asiala, Cottrill, Dubinsky,

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and Schwingendorf (1997) revealed that the concept of slope is one of the antecedents for the abstraction of "derivative of a function". Zandieh (2000) indicated that slope is one of the most important mathematical objects that should exist during abstraction process of derivative. Ubuz (2001) emphasized importance of relationship between derivative and slope to overcome difficulties about derivative. Şahin, Yenmez, and Erbaş (2015) state that relations should be constructed between derivative-rate of change, derivative-the slope of tangent line, and slope-rate of change for relational understanding of the derivative.

Despite the importance of the concept of slope is emphasized in the literature, the lack of studies about learning process of slope draws attention. The majority of the existing studies about slope have been conducted with high school or college students (e. g. Simon & Blume, 1994; Lobato & Thanheiser, 2002; Tabaghi et al., 2009; Moore-Russo et al., 2011; Duncan & Chick, 2013). Therefore, it is one of the essentials that the slope should be constructed on a solid ground since early school years in order to be able to overcome the difficulties (e. g. Barr, 1981; Teuscher & Reys, 2010; Gökçek & Açıkyıldız, 2016) in the notion of slope and derivative where slope is a prerequisite in high school and college years (Stanton & Moore-Russo, 2012). In this regard, the purpose of this study is to investigate eighth grade students' construction processes of the concept of slope.

Related Literature

According to the search of literature on the concept of slope, studies can be categorized into three types: studying the conceptualizations of slope, studies on the existing difficulties and misconceptions about slope and studies on learning-teaching process of slope. These studies, mostly conducted at high school and college level, will be presented in these three categories, respectively.

Stump (1999) was the first researcher to examine different conceptualizations of the concept of slope. Stump (1999) observed that high school mathematics teachers' definitions of slope are separated into seven groups as *geometrical ratio (rise over run), physical property (daily meaning), functional characteristics (rate of change between variables), algebraic characteristics* (y_2-y_1/x_2-x_1), *trigonometric concept (tangent), parametric relationship (y=mx+n) and the concept of calculus (relationship between derivative)*. It has become evident that addressing different representations of slope need to take place in the scope of high school mathematics teachers training. In another study by Stump (2001) on the development of pre-service teachers' pedagogical content knowledge of slope, it was aimed for teachers to have knowledge of what probable difficulties students might have in constructing the concept of slope and why they have these difficulties. In addition, Stump has added the *real world situations (either static physical situation or dynamic functional situation)* to seven different types of conceptualizations of slope. Moore-Russo et al. (2011) extended the categories of conceptualizing slope by adding three other categories, namely the *determining property (property that determines if the lines are parallel or perpendicular), behavior indicator (property that indicates the increasing, decreasing, or horizontal trends of a line)* and *the linear constant (the property which shows the lack of curvature on a line, meaning that it is a "straight" line signified with a constant).* Stanton and Moore-Russo (2012), investigated how these conceptualizations of slope were taught in the USA, emphasized the importance of the concept of slope and that the concept of slope should gain both algebraic and geometric formal meaning in eighth grade. They emphasized the importance of designing courses at this level in which students are able to establish the relationship between their initial conception of slope and its different representations. Hoffman (2015), who emphasized construction the concept of slope from the eighth grade, investigated middle school mathematics teachers' conceptions and concept images of the concept of slope. Hoffman concluded that participants mostly had conceptualizations of physical properties, geometric and algebraic ratio. Dündar (2015) conducted a similar study with 192 middle school mathematics teachers from first to fourth graders and investigated their concept images for the concept of slope. Dündar concluded that participants who are more likely to have physical, trigonometric, and geometric conceptualizations have shown that as their grade increases, the physical feature-based images for the concept of slope tend to be based on the trigonometric concept-based images.

One of the most common results of the studies that reveal the difficulties and misconceptions students have regarding the concept of slope is height-slope complexity (Leinhardt, Zaslavsky, & Stein, 1990). Clement (1985) in his research, which reveals misconceptions in graphics, found out that while students compare the slope of the straight lines constituting the hypotenuses of right triangles they have the misconception that the triangles with greater height (vertical distance) will have the greater slope as well. Barr (1981) stated that in calculus, the existing difficulties can not be overcome by emphasizing the performances that are demonstrated either operationally or mechanically. The researcher, however, emphasized the importance of constructing the concept of slope, which is basis for derivative, revealed students' difficulties related to the concept of slope. The apparent difficulties follows:

i. *Confusion with the idea that the gradient can be considered as a ratio, but a ratio of what? The gradient of the line $y = 3x + 2$ is 3, but is 3 a ratio?*
ii. *Confusion with the question of "x over y" or "y over x"? when finding a gradient of a line given two points.*
iii. *Confusion of the m (gradient) and c (intercept) given in the general equation of a straight line of the form $y = mx + c$.*
iv. *Inability to find the gradient of a line when given two points.*
v. *Inability to derive an expression for δy given two points and a function (pp.17).*

Barr (1981) and Crawford and Scott (2000) argue that difficulties can arise from the fact that students cannot understand the concept of slope as a rate of change, in other words, by memorizing rules without making relation between the rate of change and slope. Hattikudur et al. (2012) examined the complexity between "m" and "c". The researchers have focused on the information about the slope of a line and y-intercept points of 180 middle school students in the process of creating and interpreting linear functions. They revealed that students make both arithmetic errors related to slope as well as graphical errors related to interpretation of the magnitude of the slope in the graphing and interpretation process. Besides, it has been revealed that students of this age are more difficult to interpret the y-intercept of the line than the slope of the line. It is pointed out that the cause of this difficulty may be the content of the textbooks as well as the fact that the slope may be experienced informally from early years. Simon and Blume (1994) found the major difficulties that classroom teacher candidates experienced during learning the concept of slope in their study of the ratio as a measure. In the case of slope calculation, which requires the ratio to be presented as a measure, the students tend to calculate with the difference between the vertical distance and the horizontal distance. However, they have determined that this is not suitable for the slope and the ratio between vertical to horizontal distance is the correct one instead. It is stated in this study that the internalization of many different sample ramps with the same slope, which may be different in magnitude from each other, may play an important role in understanding the slope as a ratio.

In the literature, just three studies have been found focusing on the learning process of the concept of slope at the middle school level. Choy (2006) conducted a study based on the theory of variation, in which 141 junior high school students were participated. It is emphasized that noticing the vertical and horizontal distances of the part of the straight line between its any two points plays an important role in making sense of the slope. Olive and Çağlayan (2007) pointed out that the backgrounds of the participants, who were 8th grade students, related to slope play a significant role in making sense of this concept and have come to the conclusion that students may interpret the concept of slope from many different perspectives. Cheng (2010) has concluded that the problem solving skills of the participants on steepness is related to the proportional reasoning skills of the participants who were 6th, 7th and 8th grade students. Duncan and Chick (2013) have focused on how slope is perceived, analyzed and measured by preservice mathematics teachers. It is concluded that understanding measurement of steepness is related to success in linear algebra and the perception of slope as a ratio. A qualitative research conducted by Lobato and Thanheiser (2002) in an online environment showed that when highest-performing high school students were asked about the slope of a line they managed to reach to the conclusion with the "rise over run" formula and did not consider slope as a measurement, but only as a number. As a result of the study, researchers have come up with four components for understanding of ratio as a measurement: (i) determining the feature to be measured, (ii) determining which quantities will have an effect on this feature, (iii) understanding the characteristics of this measurement and finally (iv) structuring as a ratio. Tabaghi et al. (2009) aimed to investigate the effect of a teaching process which is designed on the basis of the application of Dynamic Geometry Software (DGS) on the conceptualization of slope in which APOS theoretical framework has been used to interpret the students' stages of comprehensionthe concept of slope. In this study, participants are undergraduate liberal arts and social science students, it has been suggested that DGS with the feature of dragging provides an opportunity to observe the change in slope value according to the motion of line, thus abstraction can be made at object level; as for traditional approach, conceptualization can be made only at process level.

As seen in the literature, in the beginning of the process of constructing slope as a mathematical concept, there is a need for studies conducted specifically on the conceptual development of slope during the years of middle school. Studies examining the particular processes of how students learn about slope would help us devise teaching methods through which we can find out about the underlying reasons why both teachers and students make mistakes and face certain difficulties while conceptualizing slope. In this respect, they can help us prevent the possible difficulties we can encounter in the future while constructing the concept of slope. This in turn would also allow us to learn about other concepts at the advance level, which are either directly or indirectly related to the concept of slope. The present study focuses on the construction process of the concept of slope at eighth grade.

Theoretical Framework (APOS Action-Process-Object-Schema)

APOS is a theoretical framework that enables the development of cognitive structures that can be handled as comprehension stages at the same time in the learning process of a concept. Also this theoretical framework allows us to reveal the developmental construction process of the concept. Hence, the theoretical framework of this study has been adopted as APOS.APOS which is based on Piagetian Theory of Reflective Abstraction, proves the cognitive occurrences in mind in the process of learning a concept (Dubinsky, 1991). In other words, APOS is a framework to describe how mathematical concepts are learned (Oktaç & Çetin, 2016, p. 164). APOS will be briefly described here, and for further information refer to Asiala, Brown et al. (1997) and Arnon et al. (2014). According to the APOS learning theory, an individual can deal with a mathematical situation, only through the use of mental mechanisms that allow them to construct cognitive structures that they may refer to. These mental mechanisms mentioned are *interiorization* and *encapsulation* and the cognitive structures are *action, process, object and schema* (Dubinsky, Weller, Mcdonald, & Brown, 2005). These cognitive structures are also discussed as conceptual learning levels. According to APOS theoretical framework, construction of the concept starts with actions, and then it progresses to dynamic processes by the interiorization of actions and evolves from dynamic processes to encapsulated objects (Tall, 1999). Actions are repeatable physical or mental manipulations that can transform existing objects in order to obtain new objects (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). According to Dubinsky and McDonald (2001), transformation of the objects are considered external at action stage, and the student at this stageonly know how to apply calculation in an exercise through formulation or by rote-learning. When the action is reflected and an appropriate internal proceeding is constructed, the action is interiorized into the process. An individual learns a concept at process stage may think as if s/he is applying without really presenting it (Dubinsky & McDonald, 2001). Unlike an action, a process requires an individual to keep it under control and perceive it intrinsically rather than doing it by reacting to any external stimulus

(Asiala, Brown, et al., 1997). If an individual becomes fully aware of the process and is able to perform transformations and structure those transformations, then s/he has encapsulated the process inside cognitive *object* (Breidenbach et al., 1992; Dubinsky et al., 2005). In order for the concept to be cognitive structure at object stage, another action must be applied on it so that the process in dynamic structure is seen as a static entity to which action can be applied (Arnon et al., 2014, p. 21). Finally, schema is a compatible assemblage of recalled actions, processes, objects and other schemas to deal with a new mathematical problem situation (Clark et al., 1997). A different problem situation can be reflected in the concept of development in order to help overcome the grip of it reveals the consistency of the scheme (Oktaç & Çetin, 2016, p. 175).

In this theoretical framework, determining the special mental constructions that will be developed in the learning process of a concept is called genetic decomposition. Besides, APOS theoretical framework, a research framework in which the learning processes of concepts are examined has three components, which are theoretical analysis, the designation of teaching and its application and data collection and analysis. In the theoretical analysis of the concept, researchers work to suggest the first genetic decompositon that clearly identifies possible cognitive structures based on relevant literature and experience in order to formulate how a given concept is understood (Weller et al., 2000). The teaching is designed and applied on the basis of this initialgenetic decomposition. As a result of analyzing and interpreting the obtained data, it is decided that the originally proposed genetic decomposition should remain the same or be revised.

Method

This study, which investigates the 8th grade students' construction processes of the concept of slope, constitutes a significant part of an extensive graduate thesis that employs qualitative method. In the study, qualitative research method (Strauss & Corbin, 1998) has been chosen since it is an effective method in terms of revealing the mental constructions of participants in a teaching process. In this study teaching process has in accordance with the Realistic Mathematics Education (RME) approach. RME allows students to grasp the concept of slope, which they frequently encounter on a daily basis, in a formal manner through the models they themselves develop. That is why while teaching this concept we adopted the RME approach. The aim of this study is to examine the cognitive structures of the participants during the construction process of the concept of slope, which is designed based on RME. Therefore, the data obtained have been interpreted according to APOS theoretical framework based on Piaget's learning theory. It is known that APOS theoretical framework was developed on the basis of Piagetian Theory of Reflective Abstraction in order to investigate the high school or higher level students' construction processes of the higher-level mathematical concepts (Asiala, Brown, et al., 1997; Dubinsky, 1991). In addition, it is observed that the idea of slope is constructed based on the concept of ratio, i.e., a special ratio. Examination of the construction process of the concept of slope, which starts at the eighth grade, will contribute to the field and shed light on the future studies about how the concept of slope is developed at more advanced levels.

The Initial Genetic Decomposition

When the epistemological structure of the concept of slope, which is a special ratio, is considered, the need to construct this concept at the eighth grade level, at such a level that it can be used to construct related concepts at further levels has come to the light. Therefore, in order for other actions to be applied on this concept, the process of encapsulation needs to start at 8th grade. Since the exact construction of the concept at the schema stage can be completed only after the construction of other related concepts at more advanced levels, here we examine the construction process of the concept of slope at action, process and object levels.

In this study designed qualitatively within the scope of APOS theoretical framework, researchers, first of all, have claimed the first genetic decomposition based on related literature and their experiences. This first genetic decomposition has occurred as follows.

Action. The action of finding the slope value of a given line or linear visual by finding the vertical distance and horizontal distance and dividing it by creating a right triangle that accepts all or a part of the line or linear visual as a hypotenuse. At this stage, the students cannot yet make sense of the fact that for any right triangle that accepts the line segment as hypotenuse, the ratio of the vertical distance to the horizontal distance does not change. Students only think of the slope as a concept that can be calculated with the "vertical distance/horizontal distance" rule in the right triangle model. They can calculate the slope of a line in the coordinate plane by replacing the coordinates of any two points taken on the lines with "y_2-y_1/x_2-x_1" rule. However, they fail to internalize the facts that for any two points the result of the "vertical distance/horizontal distance" ratio remains the same, and that the slope is actually identical to this constant ratio.

Process. The process of interiorizing the fact that the ratio of vertical distance to horizontal distance in right triangles, which accepts any part of the line or linear visual as a hypotenuse, is constant, and thereby making sense of the fact that slope is a constant ratio. For a straight line in the coordinate plane, the slope is organized as an algebraic ratio "y_2-y_1/x_2-x_1" by means of the differences between any two points on the coordinates from the right triangle model At this stage, students are able to infer that the algebraic ratio for any two points will remain constant.

Object. The ratios of "rise over run" and "y_2-y_1/x_2-x_1", which are interiorized as "slope" at process stage are encapsulated as an object at this stage. By connecting the concept of slope to other related mathematical concepts and different problems, students at this stage are able to reflect on the concept of slope as an object.

Participants

The participants of the research were chosen through purposive sampling (Yıldırım & Şimşek, 2005) among the 8th grade students in a public school in which one of the researchers work as a mathematics teacher. It was concluded that line equation, ratio-proportion, dependent-independent variable were the basic prerequisite concepts attained from literature and from experiences of the researchers an open-ended test was prepared. Open-ended test was applied to 16 eighth graders. The data obtained were analyzed qualitatively and it was seen that students were divided into five groups according to the similarities of their performances. The students in the first group could use only recursive strategy while expanding the number patterns. However, they could not see the dependency relationship between the two variables, nor could they express the general rule of the patterns verbally. Moreover, students failed to establish proportions, and also they could not grasp the concept of linear equation. While the second group did relatively better than the first group in terms of seeing the dependency relationship and verbally expressing this relationship as a general rule, they were not aware of the concept of a variable. It was seen that by performing simple reasoning about proportions, students could solve simple problems about proportionality that are related to their daily lives. The students in the third group were able to expand on the number patterns and could make generalizations about patterns by employing functional thinking, but they had difficulty in symbolization. It was observed that students, who had some opinions about the concept of variable, could comment on linearity, and they could recall their procedural knowledge while drawing a graph of linear equation and while solving problems about proportionality. Those in the fourth group were found to have no trouble in symbolizing the general rule of the patterns that they reached with functional thinking. In addition it was found that, students in this group, who are able to solve the problems that require establishing proportions, were not capable of seeing the fact that the ratio between the proportional quantities always remains constant. However, it was also seen that students in the fourth group could apply their procedural knowledge about the linear equation. It was seen that the students in the fifth group, who are able to solve problems by employing proportional reasoning, could reflect the stability of the ratio between proportional quantities to problem situations. In addition, it was emerged that they have both procedural and conceptual knowledge about the concept of the linear equation. More detailed information on these groups can be found in Deniz (2014). In the direction of the obtained data, for each group a participant, who is considered not to distress in communication skills, was selected. Throughout the study participants will be encoded as S1, S2, S3, S4 and S5 according to the group order they represent.

Data Collection Tool

In qualitative researches, researcher talks with the individuals of a small group and compiles their statements, gathers various documents and observes behaviors (Glesse, 2012). Data was collected via clinical interviews in this qualitatively designed study. The clinical interview first developed by Piaget, is frequently used in researches in mathematics education to discover the richness in students' minds, catch the basic activities in mind and evaluate cognitive skills (Ginsburg, 1981; Clement, 2000; Baki, Karataş, & Güven, 2002). In this study, participants were asked to solve the problems which are associated with the concept of slope during clinical interviews. Clinical interviews were conducted, during the teaching process and after the instruction. Indicators of the cognitive structures forming the genetic decomposition were questioned during the interviews by the probing questions directed. In this way, in-depth information has been gathered about whether the proposed cognitive structures at the initial genetic decomposition reveal the process of constructing the concept and at what stage the individual can construct the concept of genetic decomposition. For example, a participant who can compute the slope value by dividing the vertical distance into the horizontal distance from the right triangle model that s/he built has been encouraged to make a justification by questioning what will be the slope at a different point on the same linear visual. It was interpreted that vertical distance and horizontal distance increase and decrease proportionally with or without the help of the right triangle model indicates the process conception. In the case of a problem that does not require direct slope questioning, the participant who can regard it as a ratio and associate it with the concept of similarity on a rate basis, it was interpreted as passing the object conception. The questions addressed to the participants in the clinical interviews are given in Annex 1. In addition an appropriate environment was created for the participants to think aloud during the clinical interviews carried out throughout this research. In addition, the participants were provided with ballpoint pen and paper and given the chance to draw, write and scratch anything they thought was needed. The participants were asked not to scratch what they thought were incorrect and to rewrite what they thought were correct, thus deeper knowledge acquisition related to their cognitive processes was guaranteed. The clinical interviews were recorded by a voice recorder and during the interviews the facial expressions and gestures were noted by the researcher when necessary. Moreover, in order to minimize the risk of losing any data, the performances, which were thought to have critical importance, were transcribed immediately after the interviews. After the questions of the clinical interviews were prepared, a pilot study was carried out with three students who were not participants and had a different readiness and after the interviews were examined by two experts, it was decided that the original study can be carried out with these interview questions.

Process

This study has been carried out at 8th grade level, at a class of the public school, in mathematics lesson for a month involving the teaching process of the concept of slope with a total of 6 course hours. The literature proves that students interact with the concept of slope from the childhood and prior to giving a formal meaning related to this concept, they come to the school with a concept image (Stanton & Moore-Russo, 2012). Therefore, RME based on a principle of teaching process that allows students to present their informal knowledge and mathematize the concept on the basis of their previous experiences has been adopted. Detailed information on the teaching process has been given under the heading "teaching design" in this study. Please refer to Deniz and Kabael (2017) for more detailed information. The teaching has lasted a total of 6 course hours in periods of two hours and clinical interviews have been carried out with the participants after each of the two lessons. Clinical interviews were conducted to collect data direction to the cognitive structures constructed throughout the learning process of the concept. In addition to avoid the possible disconnect in the cognitive processing between the sessions, students were given homework assignments after each meetings. 15 clinical interviews have been carried out in total throughout the research and thus, it has been ensured that the participants' construction processes of the concept can be monitored progressively and deeper knowledge can be acquired.

The data obtained from the clinical interviews have been analyzed qualitatively by thematic analysis (Glesse, 2012, p. 255; Green et al., 2007) technique. In the thematic analysis, the researcher codes the data to search for themes and patterns in the data, and then reads all of the data coded in a similar way and tries to find what is in the essence of them (Glesse, 2012, p. 255). In this research, the themes are the themes of APOS framework, which are action, process and object. During the process of analysis, two researchers coded students' words, expressions, explanations and interpretations that are though critical. Then, the researchers came together to make a comparative analysis and to check the coherence of the findings. It has been observed that the analysis results of the researchers based on the determined common themes were found to be consistent by 93%. Some sample codes under the themes have been presented at Figure 1 and some of the student performances towards these sample codes have been presented at Table 1.

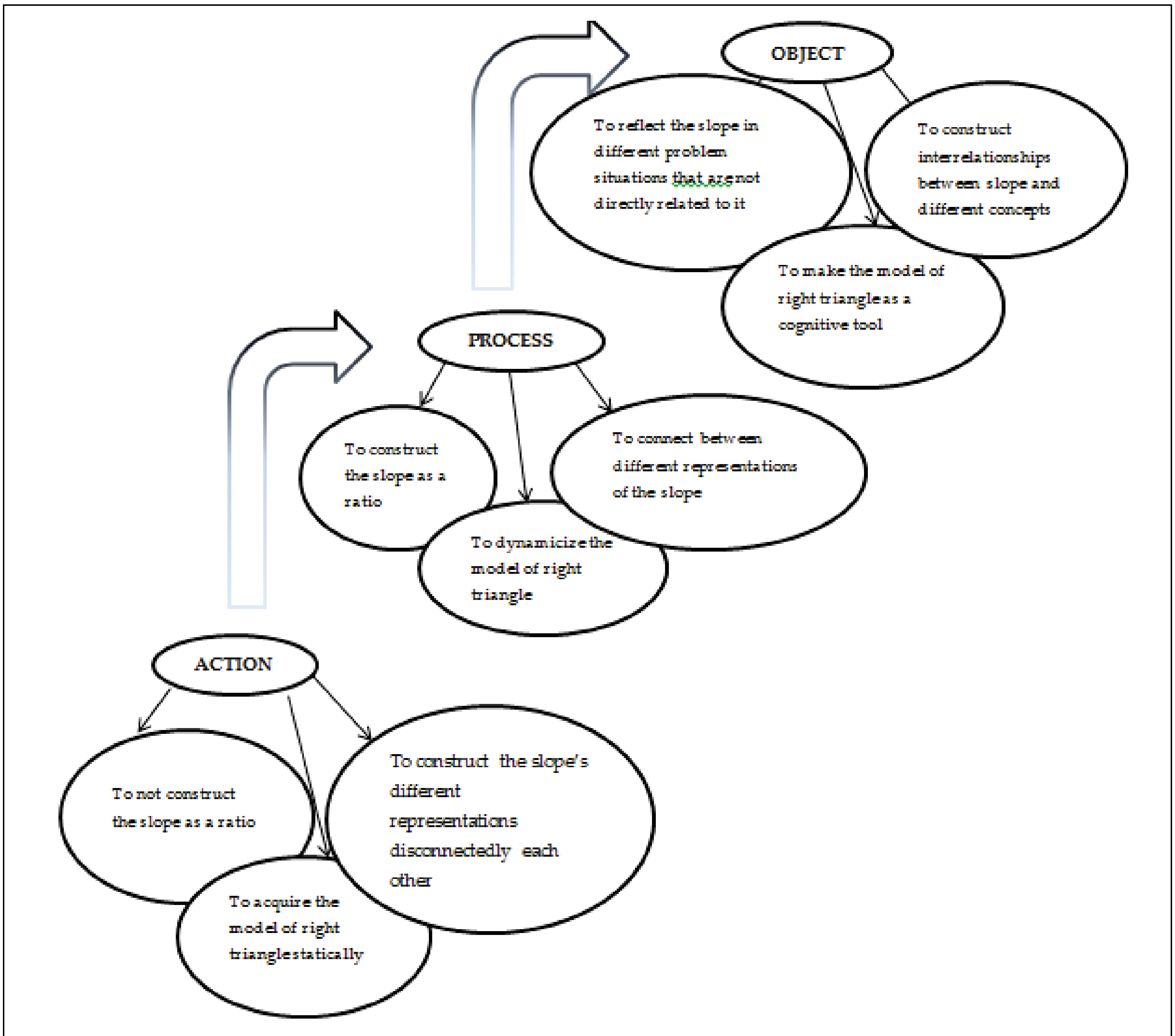


Figure 1. The Themes and Example Codes

Table 1. The Student Performances towards the Sample Codes

The Student Performances	Codes	Themes
When the heights have changed, the slope will also change.	To not construct the slope as a ratio	Action
(About the slope of a line on which any point is taken) The heights and run change. Then, the slope also changes.	To not construct the slope as a ratio	Action
(About the slope of a line on which any point is taken) The slope does not change. The ratio of the heights to the run always remains constant.	To construct the slope as a ratio	Process
(Does the slope always have to increase when the rise increases?) No. If the horizontal road changes, then the slope will change accordingly. As long as rise and run increase in the same ratio, the slope of this doesn't change.	To construct the slope as a ratio	Process
(For the algebraic ratio representation "y_2-y_1/x_2-x_1"*) I don't know why, either. We do this that way in class. That's where I've got the idea.*	To construct the slope' s different representations disconnectedly each other	Action
(For the algebraic ratio representation "y_2-y_1/x_2-x_1"*) This is the same as rise over run. We will deduct the ranges and then divide them by one another.*	To connect between different representations of the slope	Process
(In a problem situation which does not require examination of slope directly) This line segment is the same line segment and these two triangles are similar. Then, this is 3/6 (the slope of the big triangle) and this is 1/2 (the slope of the small triangle). The ratio between the edges is constant.	To reflect the slope in different problem situations that are not directly related to it	Object
(In a problem situation which does not require examination of slope directly) The slope does not change. Because we always get similar triangles. In other words, it increases linearly and goes on in the same angle.	To construct interrelationships between slope and different concepts	Object

Teaching Design

In teaching environments based on RME approach, the expectation is to have the maximum amount of interaction both among students and between students and teachers. That is why the significance of forming heterogeneous groups and having discussions both within groups and among different groups, through which formal and informal information and strategies are freely shared and defended, is emphasized. In light of literature review and the experiences of the researcher, it was concluded that learning about the concepts of rate-ratio, dependent-independent variables and linear equation was a prerequisite to studying slope. Based on their scores on a text with open-ended questions about these concepts, students were divided into five heterogeneous groups and the participating students, with whom clinical interviews are conducted, are put in different groups. Seven realistic context situations were prepared for the possible learning process that was shaped by thought experiments before the instruction. These contexts are envisaged to allow the concept to be networked with other structures existing in the mind. In this context, students were allowed to question and share their knowledge and strategies, information about exit and arrival points, initially only with their group members and later on with their classmates. Throughout the teaching experiment, students were not given any ready-made piece of information and they were given the opportunity to make a natural transition from informal knowledge to formal mathematics by creating an environment that allows them to do labeling themselves. In realistic context situations, they were asked to debate within the group first, and then to present their exit and arrival points in the whole-class discussions along with the reasoning and strategies they carried out. An appropriate learning environment was aimed for constructing progressively as a mathematical concept by guiding questions such as *" Which way is more difficulty?, What do you mean by saying more steeper?, How do you understand that it is steeper ?, What do you mean by higher? Do you show it by drawing? If the vertical distances (highest) the same, is the steepness always the same?, Is there anyone who can give a counter example?, Can the vertical distances be the same but the steepness (or the slope, inclination, skewness labels that come with them) be different?, Can the horizontal distances be the same and the slope be different? (After using the horizontal distance label), Are you sure that the slope will always be different when the height and the horizontal distance are different?"* On the basis of the initial genetic decomposition, learning paths were being planned and implemented to support the transition between the stages in the process of constructing the slope concept. For example, the discovering and making sense of that the slope is a constant ratio for each point on a line or a linear visual image was considered important in the transition to process conception. For this, individuals at different points of the same linear visual image have been asked to question the slope for the path in front of them. The discussion over the signboards showing the slope to be placed at these points has been expanded. In this process, teaching based on discussion and inquiry has been carried out in order to recall the informal knowledge of the invariance of slope on the same line and to discover that proportionally increasing and decreasing of the vertical and horizontal distances constitute a mathematical basis for it. However, it has been assumed that factors such as students' pre-information, characteristics, communication and interaction skills are expected to affect their learning process, but such uncontrollable factors are accepted as limitations of the study.

During the first two hours, students noticed the variables determining the slope. And with the support of visuals taken from daily lives they were also able to notice the horizontal and vertical distance and the angle variables. Moreover, they were able to tell whether the length of the road is a variable that affects the slope or not (Figure 2). Thus, students were supported to make sense the slope with the variables which are dependent on the physical characteristics and real-life situations of the slope.

Figure 2. Contextual Visuals that Allow the Concept of Slope to be Recalled from Daily Life and to Recognize the Vertical and Horizontal Distances that Lead to its Differentiation

During the second two hours long meeting, students were asked to interpret slope in relation to judgments about the rate of vertical to horizontal distances. Then, it was aimed the students can recall that the slope does not change even if the place of the point on the same linear visual changes. Students were questioned to discover that the slope is a constant ratio. For this aim, a context which can be create a need for the mathematical justification for the informal knowledge (Figure 3). The teaching process is designed in way to allow students accomplish the following. First, students could overcome the cognitive dissonance created by the thought that the slope cannot change even if both the vertical and horizontal distances change. So, they were able to notice that the change in height is directly proportional to the change in the horizontal distance, which in turn justifies why the slope remains constant. Finally, students were able to establish the relationship between this ratio and the slope. Through the activities of calculation and interpretation and with the help of homework, the geometrical interpretation of slope has been reinforced.

Figure 3. Contextual Visual that Allows the slope to be Fixed at Different Points of the Same Linear Visual and the Slope to be Constructed as a Geometric Ratio

During the last lesson, students were presented a plateau context (Figure 4). Students were aimed to make a transition to a line in the coordinate plane with the aid of a graph showing the covariation of the vertical descending height with the horizontal path. Then, students were asked to calculate and interpret the slope of a line segment visualized on coordinate plane, the slope of a line with coordinates from only two points and two points with high numerical values of coordinates and the slope of a line passing these two points respectively. With these activities it was aimed to create a learning environment that supports and guide the students to discover that the horizontal and vertical distances can be found step by step with the help of the difference between the coordinates. Following that, students were forced to make an algebraic generalization for a line in the coordinate plane and reconstruct their knowledge of slope as an algebraic ratio that they themselves came up with. The researcher, who is also the math teacher of these students, checks the homework assignments that were given at the end of each two hours long class meetings. That is why students take those assignments quite seriously.

Figure 4. The Transition Context of the Slope of a Line in the Coordinate Plane with the Aid of a Line Graph Showing the Relation between Vertical and Horizontal Distances

Results

In this section, the findings obtained concerning students' construction processes of the concept of slope will be given one by one and with direct quotations from clinical interviews in order for the cognitive processes to be presented one by one for each student. The participants sometimes have been used their own tags like height instead of rise, horizontal distances or bottom distance instead of run, steepness or gradient instead of slope in the quotations. Following that, the results obtained within the scope of APOS theoretical framework based on findings will be presented.

S1 is a participant who can calculate the slope by following the steps of algorithm, which are "find the rise and the run, and divide the rise by the run", but cannot construct the concept of slope as a ratio between rise and run throughout the study.

This participant, who have made her explanations or interpretations related to the concept of slope by using the right triangle model since the first interview, stated in the first interview that the slope changed depending on rise and run but could not interpret these changes. Although she used the right triangle model as a tool for the concept of slope and stated that the slope changed depending on rise and run, it has been observed that she had difficulties in changing the right triangle model dynamically and interpreting the change of slope depending on rise and run. For example, S1 stated that the slopes of the two pitched roads whose lengths were equal should be the same. Moreover, when she was asked to draw two roads whose rises were the same but slopes were different, she stated that she should shorten the run, but could not visualize this and minimized the right triangle model by shortening rise and run (Figure 5). Some of the quotations from the first interview of S1 have been given as follows.

I: *Now let's assume that there are two roads whose lengths are equal. What can you tell about the steepness of these two roads?*

S1: *If the lengths of both of the roads are the same, the steepness is also the same.*

….

I: *Okay, if I tell you to keep the heights the same, but one of the roads is steeper, what do you do?*

S1: *I shorten the run.*

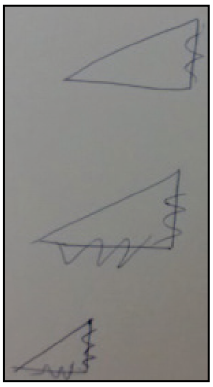


Figure 5. S1's Drawings for the Road Models whose Rises are the Same

S1, who had had difficulties in interpreting the change in slope by changing the right triangle model dynamically in the first interview, has been observed to be able to calculate the slope by applying the steps of algorithm, which are "find the rise and the run, and divide the rise by the run" in the second clinical interview. Moreover, it has been observed that she could not interiorize the change of slope according to the changes of rise and run in the second interview. It has been observed that S1 thought that the slope would change as rise and run changed according to the point taken on a linear visual given to S1 in the first interview and was not aware of the fact that the slope would not change and frequently thought that slope was dependent on the rise.

I: *Does the steepness change when the point on the road has changed?*

S1: *As it goes up, the gradient may get higher.*

I: *Well, does the steepness of the road change for this road as it goes up?*

S1: *Yes.*

I: *Why?*

S1: *When the height changes, it also changes.*

S1, who emphasized that *"because the height has changed, the slope will also change"* on the different points taken on a line, has been observed to have still interpreted the slope depending on the rise and not to have been in the process of constructing the slope as a ratio in the last clinical interview (Figure 6).

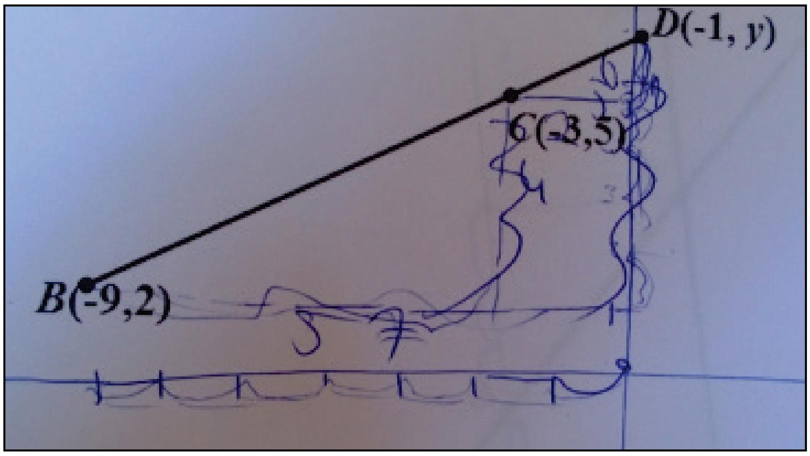


Figure 6. S1's Interpretation according to the Points Taken on the Same Line

I: *Let's say you are walking on this line. You will pass through these three points. How do you interpret the slope of the line according to these points?*

S1: *The slope will change. Because its height has changed, the slope will also change.*

Moreover, when this participant was asked to find the slope of a line that passes through these points by giving the coordinates of these two points, she was able to find the rise and the run from the differences between the coordinates without visualizing them on the coordinate plane. It has been thought that this participant could also apply the algorithm, which she developed, for the slope of a line given on the coordinate plane; in other words, she interiorized this algorithm as a tool for slope calculation. However, when S1 was asked as to why she had used this algorithm, she stated: "*I don't know why, either. We do this that way in class. That's where I've got the idea.*" and showed that she could not make sense of the ratio that gives the slope. The fact that this participant has always calculated the rise and the run by deducting the small number from the big number and has not got any negative result of slope depending on this later on this interview supports the fact that he cannot construct the slope as a ratio (Figure 7).

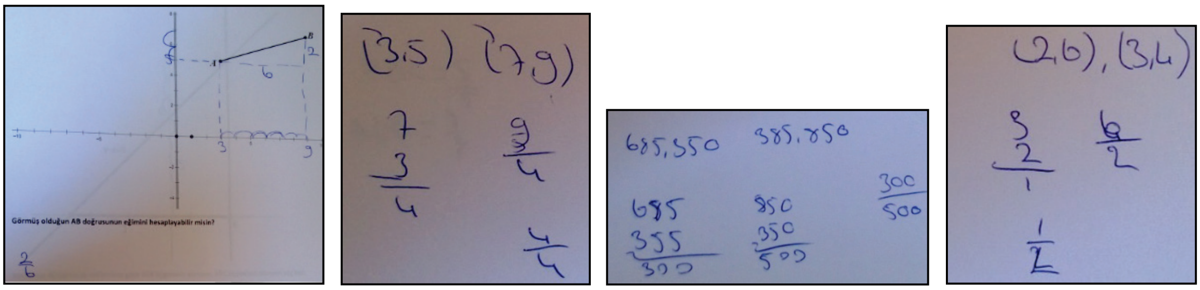


Figure 7. S1's Slope Calculations for a Line or Line Segments that Pass through the Points whose Coordinates are Known

S1, who cannot construct slope as a ratio, could not recall this concept in a problem situation in which she could not directly see it naturally. When S1 was given the components of the two points on a line, she could not use the concept of slope as a tool to find the ordinate of the third point given the only abscissa on the line.

S2's Construction Process of the Slope

S2 is a participant who has succeeded in constructing the slope as a ratio by interiorizing the proportional relationship between rise and run.

S2 could not only describe the variables on which slope is dependent, but also interpret the slope by relating to them in the first interview carried out. Furthermore, it has been observed that he could use the right triangle model dynamically in the interpretation process, and realize and correct when he made a mistake. When he was asked to draw two roads whose slopes were different S2, who used the right triangle model, constructed models whose rises were different, but runs were the same and reported that he was aware of the invariables and variables when interpreting the slope (Figure 8).

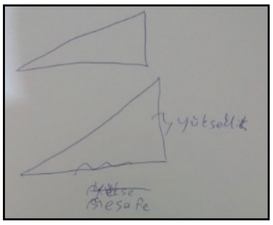


Figure 8. S2's Model of Right Triangle whose Slopes are Different

I: *Why do you say that the second road is steeper?*

S2: *Because its height is more than the other and the runs are equal.*

It has been observed that when he was asked to draw two roads whose rises were the same but slopes were different, both the rise and path length (hypotenuse) were the same in both of the right triangle model. However, the fact that he argued by using the models that the runs could not be different in that case and the path length should also be different, besides the fact that he emphasized that when the rises were the same, the slope of the one whose run was shorter had the higher slope and by using the models, he could lengthen and shorten the variables dynamically and therefore, interpret the slope (Figure 9). Whereas, S1 has always performed the horizontal distance in the direction of the shortening when she shortens the vertical distance. Moreover, she regards only the vertical distance for the slope at different points on the same line without seeing the proportional increase of the vertical and horizontal distances and she said that "the slope also changes if the height changes".

I: *Can you draw two roads on which the heights are the same but the slope is different?*

S2: *The runs should be different in that case.*

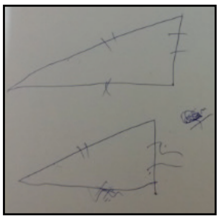


Figure 9. S2's Model

I: *Then, which one of these two is steeper?*

S2: *I've made a mistake here, haven't I, teacher? There*

I: *Why? Why do you think you've made a mistake?*

S2: *The heights are equal there. In that case, this (he is talking about the hypotenuse of the first figure whose run is longer) should be longer.*

I: *Which one do you say is steeper?*

S2: *The shorter road is steeper.*

S2, who could calculate the slope by proportioning rise and run, has been observed to have stated that the slope would not change even if a different point was taken on the same line. However, the fact that he interpreted the invariability of slope with the ratio of hypotenuse to the run, not with the ratio of the rise to the run and stated that the rise remained constant according to the point has given rise to the thought that he had misconceptions in the process of constructing the slope as a ratio between rise and run.

I: *Does the numerical value (slope) that you will write on the board change if the road is halved or reduced to its one-third, or not?*

S2: *It doesn't change.*

I: *Why doesn't the slope of road change?*

S2: *Because the height is constant. The road has decreased as the bottom distance has decreased. This and that have decreased in the same ratio (he is talking about the run and hypotenuse).*

However, S2 has been observed to have been able to explain the invariability of slope according to the point taken on the same line with the ratio between rise and run in the last interview. However, in similar situations S1 demonstrated she could not construct the slope as a constant ratio by saying that: "if the height changes, the slope also changes".

S2: *How can I explain? The slope doesn't change, but the distance rises and decreases. For example, the slope is constant. It doesn't change. The height has decreased and the bottom distance has also decreased in the same ratio.*

Furthermore, in order to calculate the slope of a line passing through the two points given, he needed to visualize the line passing through those points on the coordinate plane and could reach to the conclusion by calculating rise and run by means of constructing the right triangle model on the coordinate plane. It has drawn attention that unlike the participants who considered *"rise over run"* or the generalization of "y_1-y_2/x_1-x_2" as an algorithm they used in the calculation, even when S2 was given the points having the high-valued coordinates, he tended to calculate the slope by visualizing (Figure 10).

I: *We are given another two points and asked the slope of a line passing through these two points. Write down the coordinates. (685,350) is our first point. Our second point is (385, 850). (The student is taking notes). Can you calculate the slope of a line passing through these two points?*

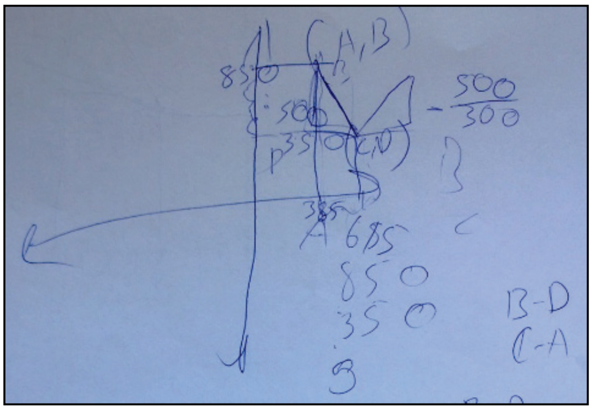


Figure 10. The Slope of a Line Passing through these Two Points which have the High-Valued Coordinates

Moreover, this participant, who realized that the result of the slope might be negative, could explain the negativity visually by its being oblique to the right or left. When S2 was asked to construct an algebraic statement appropriate to the things he had done for the slope of a line on the coordinate plane, he also succeeded in doing this and showed that he could use the algebraic statement he had got for slope calculation in the last interview. When S2 was asked to find the ordinate of the third point given the only abscissa on a line whose two points were given, he could recall the slope only with the external support from the interviewer and stimuli.

S3's Construction Process of the Slope

S3 is another participant who can construct the slope as a ratio. S3, who seemed to have been able to use the right triangle model dynamically since the first interview, has showed that she not only interpreted the slope depending on rise and run, but also started to use the angular relationship. For example, the fact that when the participant, who seemed to have kept the run constant while drawing two roads whose rises were different, was asked to keep the rise constant, she changed the runs to change the slope has drawn attention. It has been observed that S3, who stated that "*If the heights are the same, we look at the runs of the two roads*", considered the angles as well in the meantime and started to associate them with the slope. The fact that she stated that "*The heights here are the same, but its angle tightens. The one with the tightened angle is steeper*", meaning the rise on the model and the angle between hypotenuse has been considered important in terms of her starting to interrelate between slope and angular relationship. When she was asked to draw two roads whose not only rises but also path lengths were the same but slopes were different, the fact that she stated "*That's not possible...the slope has to be the same in that case*" has given rise to the thought that she could rotate, enlarge or move the right triangle model dynamically in mind and depending on this, could interpret the slope according to any variable she wanted. Similarly, in the same way, S2 expressed that the slope should remain the same in such a case and defended this view in the right triangle model. However, S1, indicating that the right triangle model can also be used as a tool to calculate the slope at the same time, could not establish a proportional relation and associate the slope with the constant ratio, although she was trying to defend that the vertical and horizontal distances affects the slope on this tool.

S3, who could calculate the slope of a linear visual given in the second interview by dividing the rise by the run, has first argued the fact that slope would not change according to the point taken on the same line by stating "*the same road, the same gradient*". Then, S3 has proven to have had an imbalance between the information from the informal life and the information from the mathematics in which she tried to construct by stating "*Because rise and run will change, the slope will also change as we divide the rise by the run*". S3 realized that the slope would not change when arguing through numerical examples, and structured to make sense of the fact that the slope would remain constant because the ratio between rise and run remained constant during this interview.

> ***S3:** Its lower distance is 240. And its rise until which the sign will be put is 70. We will divide 70 by 240.*
>
> ***I:** Has the slope changed?*
>
> ***S3:** It hasn't. When looked at the results here, the slope hasn't changed. The ratio of the height to the run always remains constant.*

When she was given a line segment on the coordinate plane in the third interview, S3 who seemed to have calculated the slope by the algebraic ratio of "y_1-y_2/x_1-x_2" showed to have associated the algebraic and geometrical interpretations of the slope with each other by stating "*The result will be the same. We will deduct the ranges and then divide them by one another. Just like the one in this formula...*", meaning the ratio of "*rise over run*" when *asked* whether she could reach to the conclusion through a different way (figure 11). On the other hand, although S1 uses the algebraic ratio when the coordinates of the points are given on a straight line, it was seen that she has only seen it as an algorithm "subtract y coordinates to find height, subtract x coordinates to find horizontal distance and divide each other". It was seen that she could not internalize the result given by this algorithm for any two points on the straight line will always be constant. It has been pointed out that S2, who avoids using the algebraic ratio, also reaches the result with the help of "vertical distance / horizontal distance" by visualizing the straight line in the coordinate plane even if the points with large coordinate values are given. However, the fact that S1 could find vertical and horizontal distances with the help of the difference between the coordinates was interpreted as a weak link between the geometric ratio and the algebraic ratio. Explaining the algebraic ratio of S2 with its own letters and associating it with the geometric ratio can be seen as an important step in the transition from the action stage to the process stage.

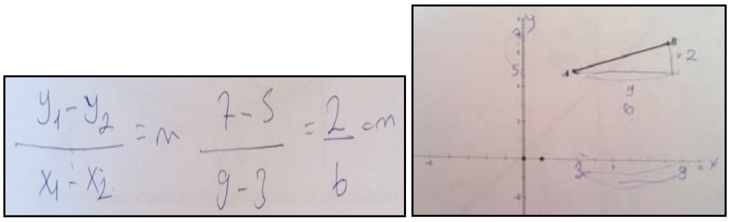


Figure 11. Images from S3's Performance on which she used the Interpretations of both Algebraic and Geometrical Ratios of the Slope

S3 managed to recall the slope with the help of the stimulant question of the interviewer and could easily calculate the slope from the known two points by emphasizing the fact that the slope would remain the same (Figure 12). S3 has shown in this step that has more robust conceptualization than S2 in terms of internalizing the invariance of the slope at any two points on the given line. Because not only defended by justifying the slope to be constant ratio but also she used it. S1, on the other hand, had shown that the slope could change when different points were taken, and therefore the slope could not be internalized as a constant ratio. Therefore, she was not to be able to apply his slope conception during the solution to this problem.

I: *What kind of conveniences does these three points' being on the same line provide to you?*

S3: *Their slopes will be the same. We deduct x from the x and we deduct y from the y (she took and divided the difference between the ordinates and abscissas of B and C points whose coordinates were known): 3/6.*

I: *What if you take another two points on the line?*

S3: *This and that (she is pointing the C and D points). We get 3/6 again because it's the same line.*

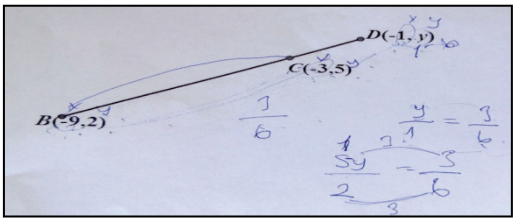


Figure 12. An Image from S3's Performance on the Question of Finding the Ordinate of the Third Point

S4's Construction Process of the Slope

S4, who has considered the slope as a formula with which she could calculate "rise over run" throughout the study, is a participant who could not construct the slope as a ratio between rise and run.

It has been observed that this participant could use the right triangle model as a tool and interpret the slope by rotating it dynamically when defending in the first interview carried out. For example, when S4 was asked to draw roads whose rises were the same and slopes were different, she showed to have been aware of keeping the runs different in the meantime (Figure 13).

I: *Can you draw two roads whose rises will be the same but slopes will be different?*

S4: *Let's say here is 5 cm and here is 5 cm (she is murmuring when drawing two right triangles). It can be drawn.*

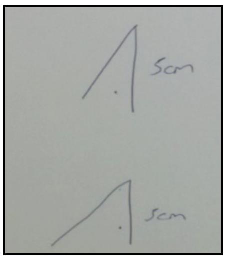


Figure 13. S4's Models of Roads whose Rises are the Same but Slopes are Different

I: *You say the heights are the same and you've given 5 cm to each of them.*

S4: *This one is steeper (the first model).*

I: *Why?*

S4: *Because the distances here are different (she pointed to the runs).*

On the other hand, it has been observed that S4 proportioned the run to the rise instead of proportioning the rise to the run when calculating the slope in the second interview. When he was confronted with the questions providing an opportunity to realize her mistake on the models she had drawn, the fact that she knew accurately on which the slope would be higher, but thought that it might be normal for this comparison to give an exact opposite result when calculated mathematically has drawn attention (Figure 14).

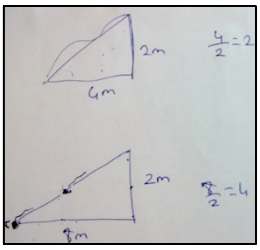


Figure 14. S4's Misconception of the Slope Calculation by Dividing the Run by the Rise

I: *So, which one has the higher slope?*

S4: *This one has the higher slope...Wait! This one has the higher slope (with excitement, she chose the one whose slope was higher visually). Because slope was inversely proportional to these (she is pointing to the runs).*

I: *Well, you say its slope is higher, but when we look at the results to which you've reached, do these results have to be the same or not?*

S4: *I think not. Because we've got 4 m for this. As we've got 4 m for this and it is higher, the rise decreases. We've got 2 m for here. As its meter, i.e., distance is shorter, the slope is steeper, I think. I mean it is not.*

S4 argued that the slope would change when different points were taken on the same line by stating *"the height and run change. Then, the slope also changes"*. It was seen that all groups noticed that the vertical and horizontal distance had been varying according to the point on the line at this stage of the teaching. However, in order to overcome the cognitive imbalance caused by the knowledge that the informally acquired slope should not change on the same line, it was expected to be understood that the ratio of the vertical distance to the horizontal distance remained constant and that the slope did not change because of this constant ratio throughout intra-group and whole-class discussions. Thus, the process had been progressed in this way and it was seen that S2 and S3 could internalize that the ratio has not been changed, although the mathematically vertical and horizontal distances have changed. However, it is thought that it was effective in the case of not achieving the desired conceptualization of S4 that she could not make sense the ratio between the linearly proportional variables in the open-ended test in addition to the uncontrollable variables. As a matter of fact, this participant had some difficulties in interpreting the slope according to the change in vertical and horizontal distance at the first interviews. On the second interview, it was seen that she had misconceptions during the construction process of slope as a ratio. S4, who seemed to have calculated the slope by proportioning the run to the rise throughout the second interview, was observed to have started to divide the rise by the run when calculating the slope and calculate the slope always by visualizing when calculating the slope of a line whose coordinates of the two points were given in the third interview (Figure 15). Moreover, S4 who seemed to be unaware of the negative slope, stated that she had never seen a formula like y_1-y_2/x_1-x_2 and showed that she had not transited from the geometrical ratio interpretation of the slope to the algebraic ratio interpretation. This participant showed that she used the right triangle model as a tool by using the ratio of "rise over run" for a line on the coordinate plane. However, the model of S4, who could not see that the rise and run decreased and increased on a line in the same ratio, has been thought not to have been dynamized entirely yet.

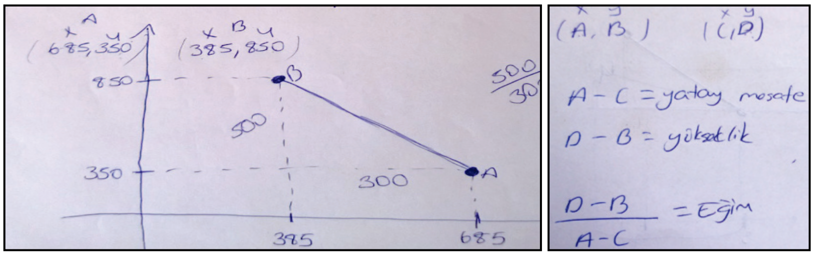


Figure 15. S4's Calculation of the Slope of a Line Passing through the Two Points whose Coordinates are Given

> ***S4:*** *When you give me two points, I find where these two points are. For example, if the first point you have given is point A, I find the point A; if the second point you have given is point B, I find the point B. We have shown them on the coordinate plane. I find the height and run there. I write the heights on top, I calculate the slope from rise over run.*

In a question in which the ordinate of the third point given the only it's abscissa on a line whose two points were given and there was no direct questioning of the slope, S4 could not recall the slope and even if she could recall it with the help of the interviewer, she could not make sense of the invariability of slope on the same line and could not reach to the conclusion of the question.

S5's Construction Process of the Slope

S5, who have been concluded to have constructed the slope as a ratio, is a participant who has been thought to have interrelated between the concept of similarity and slope, perceived the relationship between slope and angle, rotated, enlarged or moved the right triangle model dynamically without needing to present it psychically and succeeded in interiorizing it as a cognitive tool. Moreover, this participant, who seemed to have constructed the slope also as an algebraic ratio, have been observed to have managed to make dynamic transitions between the geometrical ratio and algebraic ratio interpretations within the same question and succeeded in recalling and using the slope without any external support or stimulus in different problem situations which did not require any question of direct slope.

It has drawn attention that S5, who seemed to have managed to interpret the slope very easily according to the variables to which the slope was related since the first interview, could rotate, enlarge or move the right triangle model dynamically. It has been observed that when she kept the variable constant, she could explain the relationship between slope and the other variable by stating "*As the height increases, the steepness also increases. As the run on the horizontal road decreases, its steepness increases again*" or "*The highest and slope are directly proportional. And it is inversely proportional to the run made on the horizontal road.*" It has also been observed that she could make sense of the fact that slope is not directly related to the line or the length of the linear visual because rise and run increase and decrease in the same ratio.

> ***I:*** *Okay, does the slope always have to increase when the rise increases?*
>
> ***S5:*** No. *If the horizontal road changes, then the slope will change accordingly. As long as height and run increase in the same ratio, the slope of this doesn't change.*

It has been observed that she stated "*when it comes to slope, I think it is perpendicularity or obliquity generated depending on the edges of something*" and could interpret the right triangle model according to the rise and run by picturing it in mind. It has been observed that the participant, who did not seem to have had difficulty in calculating the slope of a linear visual given, emphasized that she could minimize the right triangle model as much as she liked in the process of constructing the right triangle model appropriate to the slope of the line she had calculated because the slope would not change. S5 demonstrated that she was in process of interiorizing of the slope as a ratio by explaining the height of the peak point when the slope of ridgeway and horizontal distance were given.

***S5:** Horizontal distance is 1000m. Slope is 3/10. As if the horizontal distance was said 10. 1000 is 100 time 10. Therefore we should take 100 times 3. So the height is 300.*

It has been observed that the participant, who has driven attention to have recalled and used the slope in the process of solution without receiving any external support or stimulus for both of the questions which did not contain any questioning of direct slope in the third interview, started to interrelate between the concepts of similarity and slope (Figure 16).

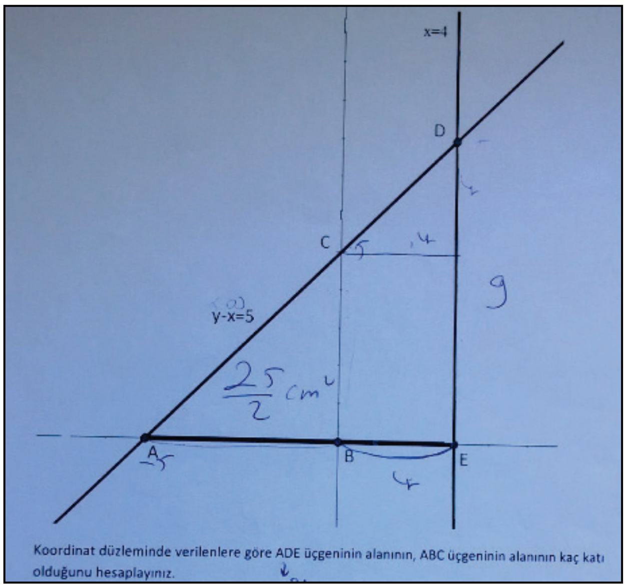


Figure 16. An Image from S5's Performance when Proportioning the Areas of the Two Triangles

***S5:** The slopes of these two are equal.*

***I:** Are the slopes of them equal? Whose slopes are equal?*

***S5:** The slopes of the hypotenuses of these two triangles are equal. After all, if we draw a triangle here (the triangle regarding DC as hypotenuse), here is 4 (she is pointing to the edge across the BE) and from here we get 1 (she is showing that the ratio of similarity is 1 by writing) and as the two triangles are similar, the slope doesn't change anyway.*

S5, who seemed to have managed to explain the constant state of the slope on the same line visually (with the expression of the line's not zigzagging) and with the linear increase in the values of rise and run and with the angle's remaining constant, has showed to have constructed the slope by interiorizing it with different interpretations (Figure 17).

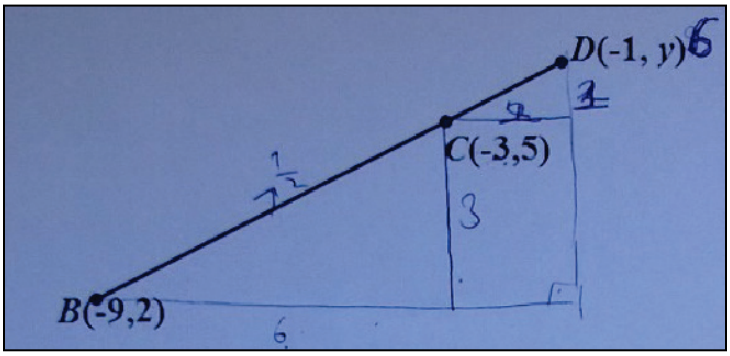


Figure 17. An Image from S5's Performance on the Question of Finding the Ordinates of the Three Points on the Same Line

***I:** How do you know the slope hasn't changed on this road?*

***S5:** This line segment is the same line segment and these two triangles are similar. Then, this is 3/6 (the slope of the big triangle) and this is 1/2 (the slope of the small triangle). The ratio between the edges is constant.*

***I:** For example, if we lengthen this line a little more, will the slope not change?*

***S5:** It will not change. It will not change as long as we don't draw zigzags.*

***I:** Why doesn't the slope change?*

***S5:** Because we always get similar triangles. In other words, it increases linearly and goes on in the same angle.*

The participant, who seemed to have found the slope of a line passing through the two points given also with the algebraic generalization of y_2-y_1/x_2-x_1 without visualizing it on the coordinate plane, has been observed to have been aware of all the steps she had taken and explained how she had reached that generalization (Figure 18).

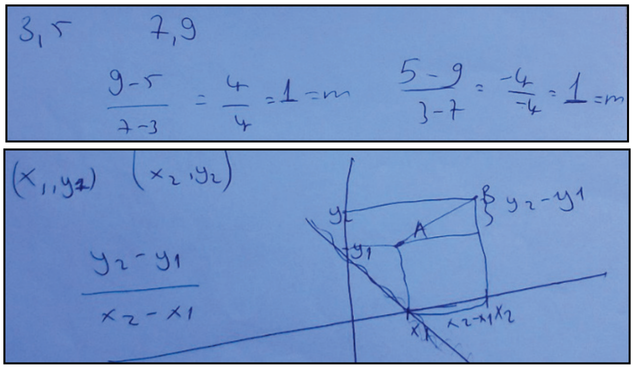


Figure 18. S5's Explanation of the Process of the Calculation of the Slope and Reaching Generalization by the Algebraic Ratio Interpretation with the Model

It has become evident that S5, who seemed to have been aware of the possibility of the slope's being negative and also managed to reach to the conclusions of negative slope, she reached to a generalization by stating "*the negative lines are always oblique to the left and the positives are always oblique to the right or the negatives have wide angles and positives have acute angles*". It has been observed that S5, who has drawn attention to have managed to calculate the slope of a line given both by the ratio of rise to the run from the right triangle model and by the generalization of y_2-y_1/x_2-x_1 on the coordinate plane, and made a meaningful transition from the geometrical ratio interpretation of the slope to the algebraic ratio interpretation accordingly, could explain how to calculate the slope of a line, only the equation of which was given. It has been observed that when she was asked whether she could calculate the slope of the line without drawing it on the coordinate plane or not, she reached to the rise and run from the points on which the line crosses the axes of x and y, and reached to the conclusion from the geometrical ratio interpretation. In the meantime, she showed not to have needed to present the right triangle model physically but by picturing it in mind (Figure 19). In the meantime, it was seen that the visualization of the right triangle model in the mind was used in a rapid and serial manner, not step by step. Even it has been pointed out that the participant was not aware that she was using the model. For this reason, it was interpreted that S5 has integrated the right triangle model to the slope conception as an inseparable entity.

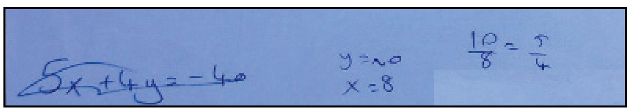


Figure 19. S5's Slope Calculation of a Line whose Equation is Given

The fact that S5, who suggested that she was aware of the fact that she could take any two points other than the points crossing the axes on the line, she could calculate the slope by making use of the algebraic ratio interpretation this time has drawn attention. It has been observed that S5, who seemed to have managed to use the different interpretations of the slope for the same question, not only adopted the generalizations related to slope by making sense of them, but also could use them in a meaningful manner in problem situations by interiorizing them in the meantime (Figure 20).

S5: *I shall use 5x+4y-40=0. I've given 1 to y. It is now -36. When the x is crossed, it is now 36/5 (she is telling as she is calculating on the paper). Then, I've given 1 to x. When we've given 1 to x, it's 4y=35. And y=35/4. Here the coordinate is 36/5. It is now (36/5, 1). Then here, what have I given to x? I've given 1. It is now (1, 35/4). I will deduct one of them. I will deduct 35/4 from 1, and then I will deduct 36/5 from 1 (she is showing by writing). Then I will divide.*

I: *Okay, divide it.*

S5: *Here it will be -31/4 (she finds the dividend by operating). And here will be 31/5 (she has found the dividend by operating). (She has written both of them as divisions) If we reverse this (she is talking about the denominator), the result will be -5/4 again.*

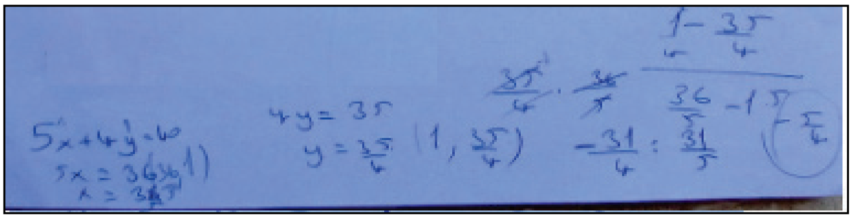


Figure 20. S5's Slope Calculation of a Line whose Equation is Given with the Algebraic Ratio Interpretation

Conclusion

In this section, the results obtained from the findings of the study according to the cognitive structures in the APOS learning theoretical framework and the genetic decomposition revised in line with these results will be presented. In line with the findings obtained, the constructions of the geometrical and algebraic representations of slope in the genetic decomposition have been found necessary to be discussed independently. In this view it is thought that the construction process of the concept can be described more clearly. In the obtained findings although the individual's slope conceptualization as a geometric constant ratio does indicate the process stage, this performance does not require to be at the same stage to the conceptualization of the algebraic ratio. For example, in the conceptualization of the geometric ratio, it was thought that S2 could internalize the slope on the same line as a constant ratio, but in the conceptualization of algebraic ratio, it was seen that he could not internalize "y_2-y_1/x_2-x_1" or similar generalization, instead, he was able to calculate and interpret the slope only by geometric ratio. Moreover, as it has been concluded that the right triangle model which accepts the whole line segment or any piece of the line whose slope is expected to be calculated or interpreted as a hypotenuse obtains different functions according to the cognitive structures constructed, the development of this model of right triangle has also been added to the genetic decomposition for slope.

1. Action

It has been observed that the students, who could construct the concept of slope at action level, could not construct it as a ratio. The students at this level have calculated the slope of a line or linear visual given via an algorithm with which they divide the rise by the run, or considered "rise over run" as a formula and reached to the conclusion by putting the values of rise belonging to that line or linear visual and run on the formula. An individual, who could construct slope at action level, could not conceptualize the slope on the basis of the proportional relationship between rise and run. In other words, s/he could not interrelate to the relationship of equality between the invariability of the slope in any point on the same linear visual and the invariability of the ratio between rise and run. Therefore, it has been observed that an individual, who has considered the slope as an algorithm with which s/he would divide the rise by the run, could not make sense of the fact that the slope would not change according to the point taken on the same line or linear visual. It has been concluded that the S1 and S4 participants could construct the concept of slope at action level yet. For example, it is seen that S1 argued that the slope would change according to the point taken on the same line or linear visual because the values of rise and run changed. It has drawn attention that S4, who could not realize that the ratio between these values did not change as the values of rise and run changed, considered the slope as an algorithm she developed by stating "find rise and run and divide them by one another". During their questioning related to their interpretations of slope according to the point taken on the same line or linear visual, both S1 and S4 have been observed to have had difficulty in thinking about it without calculating in each step. It has been concluded that these students having the construction of the concept at action level could not rotate, enlarge or move dynamically the right triangle model developing automatically in the learning process of the students and could not use it as a cognitive tool independent from the situation. Moreover, it can be seen that the right triangle model used in this stage cannot be constructed as a hypotenuse of any part of a given straight line. Instead, the right triangle model, which is regarded as hypotenuse, is only considered to be based on the starting and ending points, as if all of the line is considered as a line segment. So it is interpreted as the model's stability. In addition, in the case of the contextual modeling of the linear visual (for example mountain, ramp, downhill road etc.), during the performing the right triangle model, always assuming the whole linear visual as a hypotenuse was interpreted that supports the idea of stability of the model. It has been concluded that these students having the construction of the concept at action stage could not rotate, enlarge or move dynamically the right triangle model developing automatically in the learning process of the students and could not use it as a cognitive tool independent from the situation. Moreover, these students could make use of the ratio of "y_2-y_1/x_2-x_1" as a formula in the slope calculation, but could not interrelate between this algebraic ratio interpretation and geometrical ratio interpretation (rise over run). As a result, it is thought that the first level of the genetic decomposition will be summarized as follows:

1a. Geometrical. The action of using the ratio "vertical distance/horizontal distance" or "rise over run" as an algorithm only for calculating the slope for a given straight line or linear visual. The slope is not to be construed as a constant ratio between vertical distance and horizontal distance.

1b. Algebraic. The action of using the algebraic ratio of "y_2-y_1/x_2-x_1" as a formula and reaching the conclusion by writing down the coordinates of the two given points in their place.

Static model: In the conceptualization of action, the right triangle model which accepts the whole line segment is a static tool for making sense the situation. This is meant to model more by modeling the situation (mountain skyline, road, etc.). It can be seen that while the right triangle model is constructed in this stage, any part of the line or linear visual cannot be internalized as a hypotenuse. For this reason, all the straight line is always considered as a line segment, and it is accepted as a hypotenuse.

2. Process

An individual, who could construct the slope as a ratio between rise and run, has interiorized the fact that the slope will not change according to the point taken on the same line or linear visual. The fact that the student emphasized that the slope would not change only visually has given rise to the thought that s/he might have reached to this conclusion in line with the experiences s/he had gained from his/her informal life and therefore, could not provide conceptualization at process level yet. Furthermore, it has been concluded that an individual, who memorizes the fact that the slope will not change on the same line without making sense of it, cannot have the conceptualization of process because s/he could not interiorize the fact that the slope is a special ratio. It is seen that the individual ensuring the construction of concept at process level has interiorized the fact that the ratio of rise and run remain constant although these two change according to the point taken on the same line or linear visual. Explaining the slope as a ratio by justifying the proportional change of distances taken vertically and horizontally at any point of the line was interpreted as an indication of its interiorization. However, the fact that the invariance of the slope of the line segment between any two points on the same line can be justified by the algebraic ratio is also interpreted as an interiorization pointing to the process conception.

It is seen that the right triangle model at this stage is no longer dependent on the situation and can be rotated, enlarged or moved dynamically. It has become evident that the students could use the model dynamically in their arguments, interpretations and readjusting activities while the right triangle model has become an inseparable part for the slope. However, it has drawn attention that the right triangle model at this stage could not be used as a tool without being presented and when it would be used, the need to visualize it emerged, and therefore, it has been observed that it did not become a cognitive tool by being entirely abstracted in the conceptualization at process stage yet.

Another indicator demonstrating the construction of slope at process stage is the use of the algebraic ratio of "y_2-y_1/x_2-x_1" by its being associated with the geometrical interpretation of slope. An individual at this stage can give meaningful answers to the questions as to why s/he could find the difference between the coordinates, why s/he divided the differences by one another and why the result might be negative. The fact that S2 and S3, who has been concluded to have carried out the process conceptualization, could explain the steps they had taken with their reasons and proved to have made sense of the algebraic ratio they had used has been considered as significant. The fact that they could present their own algebraic generalizations, make dynamic transitions between the algebraic interpretation and geometrical interpretation of the slope, rotate, enlarge or move dynamically the right triangle model developed and use this model as a tool for slope even though they needed to visualize it are the main distinctive performances. However, it has been observed that the participants at this stage could recall the concept of slope only in line with the external stimuli coming from the researcher in problem situations in which the slope was not directly asked and had difficulties in using the concept. As a result, it is thought that the second stage of the revised genetic decomposition can be summarized as follows:

2a. Geometrical. The interiorization of the slope as a ratio between rise and run. The making sense of the fact that the slope does not change on a point taken on a line or linear visual is because of the fact that the ratio between rise and run remains constant.

2b. Algebraic. The making sense of the fact that the slope constructed as a ratio of the rise to the run is readjusted by associating it with the algebraic ratio of "y_2-y_1/x_2-x_1" for the slope of a line on the coordinate plane. The interiorization of the idea that any two points of a line can be taken as the basis for the slope of a line with the help of the slope as constant ratio. Transitions between the different representations of slope can be made and interrelated. Each step taken when calculating the slope is realized and this awareness can be defended with necessary explanations.

The construction process of the model as a cognitive tool: The right triangle model in the process conception becomes independent from the situation and is used as a tool in the readjusting of slope, problem situations, interpretations and defending their opinions. However, the individual requires to visualize the model in order to use it at this stage.

3. Object

When the conceptualization of the construction of a concept at process stage is completed, it is expected that the process is presented as an object by being encapsulated and thus, actions can be applied on it. It can be thought that conceptualization of the concept of slope, which starts from the eighth grade in its teaching, at the object stage will occur during the learning of concepts such as the derivative where the slope concept is a prerequisite. It is thought that the indicators of object stage at 8th grade level cannot be observed explicitly. It is expected that the object conception can be observed more clearly in high school and college years. Besides this, another indicator of the construction of a concept at object level is that the object can be reflected in process form to different problem situations or during the constructions of other concepts by the de-encapsulation mechanism. Of course, in the case of a different problem, the reflection of the concept of slope does not precisely show that it was removed from its capsule and therefore constructed as an object. But in terms of pointing to possible object construction, the ability to relate participants to different concepts, such as similarity, in different problem situations that do not question direct slope reinforces their idea of being in object stage. Two problem situations, which did not require the direct examination of slope and with which the students were not assumed to have encountered before, have been given in the third clinical interview with the aim of investigating the encapsulation process of the concept of slope among the participants. It has been observed that S5, who seemed to have been able to reflect the knowledge of slope on the problem situation and reach to the conclusion easily without expecting any external support or stimuli in both of the two problem situations, could interrelate and make transitions between the different representations (physical, algebraic, geometrical, linear constant) of slope. It has been observed that this participant, who seemed to have been able to recall and reflect the slope as a process, could interrelate between the similarity and the concept of slope on the basis of ratio and present the relationship between the angle and slope when needed. Therefore, the fact that S5 could interrelate the slope with different concepts she had, and recall and use it in an another problem situation without expecting any external stimulus or support when needed has given rise to the thought that she has completed the cognitive construction of slope at process stage and entered into the encapsulation process. Furthermore, it has been concluded that the fact that this participant started to use the right triangle model as a cognitive tool by picturing it in her mind without needing any visualization is an important indicator of the fact that she was in the process of encapsulation. S5 has been observed not only to have rotated, enlarged or moved the right triangle model dynamically, but also to have been able to explain it verbally without presenting it visually and reached to the conclusion by making calculations in mind through the cognition of the model. During the interview it has been thought that she may be not aware of using the right triangle model. As a result, with the starting of teaching the concept of slope at 8th grade level, the cognitive structure of a student at this level having acquired by transiting to the object stage can be summarized as follows:

3a. The encapsulation of a process in a way that will allow other actions to be applied on it in the construction of a different concept related to the concept of slope. Presentation of the slope by de-encapsulating as a process in different problem situations without expecting any external support or stimulus. Besides, the construction of meaningful interrelationships between slope and different cognitive structures in problem situations which are not related to the slope directly.

Model as a cognitive tool: The model at object stage is in the way of being a cognitive entity and tool that is not required to be presented physically.

Discussion and Conclusion

Slope is a concept that appears in different fields of mathematics through different representations, is expected to be recalled in the construction of many higher-level mathematics concepts and depending on this, is expected to be gained in many different disciplines such as architecture, engineering, science, space science. Learning of this concept in a meaningful manner will contribute to the performance of many professions in a more qualified manner and thereby to an increase in public welfare. Therefore, the examination of the construction process of slope and studies aiming to reveal cognitive structures during this process will respond to many questions such as how the learning of concept is carried out, the difficulties that may be encountered in the process of learning, what the prerequisite information are and how they affect the process of conceptual learning. In this study, the genetic decomposition at 8^{th} grade level has been presented and significant results have been obtained about the construction process of concept.

Crawford and Scott (2000) and Barr (1981) emphasize that slope is learned rather operationally and draw attention to the need of conceptual learning. As it is seen in this study, individuals who can construct slope as a formula or algorithm are only at the action stage of the conceptual learning. An individual at this stage consider slope as a concept that they can acquire as a result of a series of operations without making sense of it. It has been concluded that the understanding of slope as a ratio, that is investigated by Lobato and Thanheiser (2002) and Simon and Blume (1994), plays a critical role in interiorising of the action to the process in this study. It has been observed that transition from action to the process stage is possible through the interiorization of the slope as a ratio between rise and run. An individual who constructs the geometrical ratio interpretation of slope in a meaningful manner will be able to make a meaningful transition to the algebraic generalization. Thus, in the process of the reconstructing of concept schema that is ongoing in high school and college years, the different representations of slope will be prevented from being learnt in a disjointed way. When considered from this point of view, the results obtained in this study support the idea of construction of meaningful relationships between the different representations of slope and thus, ensuring higher-level conceptualization that Stump (1999, 2001) and Stanton and Moore-Russo (2012) have suggested. In this study, it has been observed that S3 and S5, who reached to the process and object stages, could make dynamic transitions between the different representations of slope. The encapsulation of the process allows for different actions to be applied on it. However, at present study it is thought that there may not be many clear situations that would allow the implementation of different actions on object at the level of eighth grade. Nevertheless, it is suggested to allow for learning situations that are thought to be invoked as a process, such as recognizing the invariance of the slope, and which may require comparison action on it in the instructional design. Besides that, the possibility of reflecting the object in different problem situations as a process by de-encapsulation of it has drawn attention in the performance of S5. This student, who can reflect this concept in problem situations that are not directly related to slope and interrelate between slope and the concepts she have such as angle and similarity, has been observed to be on the way of constructing as an object. For this reason, it will be beneficial for students to be faced with problem situations they can reflect and use the concept of slope with the aim of ensuring the object conception of slope in the process of teaching. Moreover, well-planned activities that will help them interrelate between slope and different concepts will also allow the individual for the encapsulation of the process.

In this study, the process of construction of the representations of geometrical and algebraic ratio that are the two of the 11 representations of slope that Stump (1999, 2001) and Moore-Russo et al. (2011) has presented. It has been seen that reflecting the informal conceptualizations about the slope (the physical property, the real world situations, the linear constant) in teaching environment and ensuring the conceptual development by establishing relations with them affect these two formal conceptualizations of slope positively. The construction of other representations, which are parametric coefficient, functional feature, trigonometric concept and the concept of calculus that will be introduced in high school and college years, in other words, the examination of reconstructing process of slope is deemed necessary. These researches are significant in terms of the exact presentation of the genetic decomposition of slope expected to be reflected in the construction of many mathematical concepts such as derivative.

In the researches based on APOS theoretical framework, it has been underlined that the genetic decomposition of a concept is not only expected to be presented by a research, but also the genetic decomposition is required to be supported by many studies related to it (Asiala, Brown, et al., 1997). For this reason, it is deemed necessary to re-examine the genetic decomposition that has been revealed in this study at further levels and in different researches that has been conducted on a greater sample.

Lastly, the fact that the right triangle model in this study has acquired different functions at different stages of the phases of conceptual learning has drawn attention. It has been observed that the model, which individuals make sense at action stage has been dynamisized independent from the situation at process stage and is a tool required to be presented physically. It has been observed that the model is a cognitive tool that is not required to be presented physically anymore and is a cognitive entity that an inseparable part of the schema of slope in individuals thought to be on the course of ensuring conceptualization at object stage. The conduct of detailed researches that will prove the role of models in the learning process of the concept will be an important step for answering the question as to how conceptual learning can be performed more productively.

References

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory – A framework for research and curriculum development in mathematics education*. New York: Springer.

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1997a). A framework for research and curriculum development in undergraduate mathematics education. *MAA NOTES*, 37-54.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997b). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior*, *16*(4), 399-431.

Baki, A., Karataş, İ., & Güven, B. (2002, September, 16-18). *Klinik mülakat yöntemi ile problem çözme becerilerinin değerlendirilmesi*. Paper presented at V. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, METU, Ankara.

Barr, G. (1981). Some student ideas on the concept of gradient. *Mathematics in School*, *10*(1), 14-17.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247-285.

Cheng, D. S. (2010). *Connecting proportionality and slope: Middle school students' reasoning about steepness* (Unpublished doctoral dissertation). Boston University, Boston.

Choy, C. (2006). *The use of variation theory to improve secondary three students' learning of the mathematical concept of slope* (Unpublished doctoral dissertation). University of Hong Kong, Pokfulam, Hong Kong SAR. Retrieved from http://hub.hku.hk/bitstream/10722/51380/6/FullText.pdf?accept=1

Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St John, D., & Vidakovic, D. (1997). Constructing a schema: The case of the chain rule?. *The Journal of Mathematical Behavior*, *16*(4), 345-364.

Clement, J. (1985). Misconceptions in graphing. *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education*. The Netherlands.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). London: Lawrence Erlbaum Associates.

Crawford, A. R., & Scott, W. E. (2000). Making sense of slope. *The Mathematics Teacher*, *93*(2), 114-118.

Deniz, Ö. (2014). *Examination of 8th grade students' construction of the concept of slope based on realistic mathematics education in APOS framework* (Unpublished master's thesis). Anadolu University, Institute of Educational Sciences, Eskişehir, Turkey.

Deniz, Ö., & Kabael, T. (2017). Students' mathematization process of the concept of slope within the realistic mathematics education. *Hacettepe University Journal of Education, 32*(1), 123-142. Retrieved from http://www.efdergi.hacettepe.edu.tr/upload/files/1724denizslope-mathematization-1.pdf

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-126). Dordrecht: Kluwer.

Dubinsky, E. D., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. *New Icmi Studies Series, 7*, 275-282.

Dubinsky, E., Weller, K., Mcdonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: An Apos-Based analysis: Part 1. *Educational Studies in Mathematics*, *58*(3), 335-359.

Duncan, B., & Chick, H. L. (2013). How do adults perceive, analyse and measure slope?. *In 36th annual conference of the mathematics education research group of Australasia (MERGA 36)* (pp. 258-265). Melbourne.

Dündar, S. (2015). Knowledge of mathematics teacher-candidates about the concept of slope. *Eğitimde Kuram ve Uygulama, 11*(2), 673-693.

Ginsburg, H. P. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For The Learning of Mathematics*, *1*(3), 4-11.

Glesse, C. (2012). *Nitel araştırmaya giriş* (A. Ersoy, & P. Yalçınoğlu, Trans.). Ankara: Anı Publishing.

Gökçek, T., & Açıkyıldız, G. (2016). Matematik öğretmeni adaylarının türev kavramıyla ilgili yaptıkları hatalar. *Turkish Journal of Computer and Mathematics Education, 7*(1), 112-141.

Green, J., Willis, K., Hughes, E., Small, R., Welch, N., Gibbs, L., & Daly, J. (2007). Generating best evidence from qualitative research: The role of data analysis. *Australian and New Zealand Journal of Public Health*, *31*(6), 545-550.

Hattikudur, S., Alibali, M. W., Prather, R. W., Knuth, E. J., Asquith, P., & Nathan, M. (2012). Constructing graphical representations: Middle schoolers' intuitions and developing knowledge about slope and y-intercept. *School Science and Mathematics, 112*, 230-240.

Hoffman, T. W. (2015). *Concept image of slope: Understanding middle school mathematics teachers' perspective through task-based interviews* (Unpublished doctoral dissertation). The University of North Carolina at Charlotte.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*, 1-64.

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as measure as a foundation for slope. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions: 2002 yearbook* (pp. 162-175). Reston, VA: National Council of Teachers of Mathematics.

Moore-Russo, D., Conner, A., & Rugg, K. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational studies in Mathematics*, *76*(1), 3-21.

Nagle, C., & Moore-Russo, D. (2014). The concept of slope: Comparing teachers' concept images and instructional content. *Investigations in Mathematics Learning*, *6*(2), 1-18.

Oktaç, A., & Çetin, İ. (2016). *APOS teorisi ve matematiksel kavramların öğrenimi.* E. Bingölbali, S. Arslan, & İ. Ö. Zembat (Eds.), *Matematik eğitiminde teoriler* (pp. 163- 182). Ankara: Pegem Akademi.

Olive, J., & Çağlayan, G. (2007). 8th grade students' understanding of slope and its antecedents in a learning situation based on quantitative reasoning. In D. K. Pugalee, A. Rogerson, & A. Schinck (Eds.), *Proceedings of the ninth international conference of the mathematics education into the 21st century project: Mathematics education in a global community* (pp. 491-496). Charlotte, NC: University of North Carolina at Charlotte.

Şahin, Z., Yenmez, A. A., & Erbaş, A. K. (2015). Relational understanding of the derivative concept through mathematical modeling: A case study. *Eurasia Journal of Mathematics, Science & Technology Education*, *11*(1), 177-188.

Sandoval, C. (2013). Transcript of History of Mathematics: Slope of a Line. Retrieved from https://prezi.com/8q5czopcujx9/history-of-mathematics-slope-of-a-line/

Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *The Journal of Mathematical Behavior*, *13*(2), 183-197.

Stanton, M., & Moore-Russo, D. (2012). Conceptualizations of slope: A review of state standards. *School Science and Mathematics*, *112*(5), 270-277

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Grounded theory procedures and technique* (2nd ed.). NewburyPark, London: Sage.

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal*, *11*(2), 124-144.

Stump, S. (2001). Developing preservice teachers' pedagogical content knowledge of slope. *Journal of Mathematical Behavior*, *20*, 207-227.

Tabaghi, S. G., Mamolo, A., & Sinclair, N. (2009). The effect of DGS on students' conception of slope. *In proceedings of the 31st annual meeting of the North American cahpter of the international group for the psychology of mathematics education* (Vol. 5, pp. 226-234). Atlanta, GA: Georgia State University. Retrieved from http://www.pmena.org/2009/proceedings/ALGEBRAIC%20THINKING/algebraBRR369496.pdf

Tall, D. O. (1999). Reflections on APOS theory in elementary and advanced mathematical thinking. O. Zaslavsky (Ed.), *Proceedings of the 23rd conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 111-118). Haifa, Israel.

Teuscher, D., & Reys, R. E. (2010). Slope, rate of change, and steepness: Do students understand these concepts?. *Mathematics Teacher, 103*(7), 519-524.

Ubuz, B. (2001). First year engineering students' learning of point of tangency, numerical calculation of gradients, and the approximate value of a function at a point through computers. *Journal of Computers in Mathematics and Science Teaching*, *20*(1), 113-137.

Weller, K., Clark, J., Dubinsky, E., Loch, S., McDonald, M. A., & Merkovsky, R. (2000). An examination of student performance data in recent RUMEC studies. *Washington, DC: Mathematical Association of America*. Retrieved from http://www.math.kent.edu/~edd/Performance.pdf

Yıldırım, A., & Şimşek, H. (2005). *Sosyal bilimlerde nitel araştırma yöntemleri* (5th ed.) Ankara: Seçkin Publishing.

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-127.

Appendix 1. Clinical Interview Questions

1st Clinical Interview Questions

Picture 1

Picture 2

The motorcycle you see is powered by electricity and has a certain charging capacity. The same person is planning to go with the same motorcycle in the ways you see in the picture. The motorcycle will be charged before each road goes out. Which way do you think the motorcycle charge ends quickly?

Why?

What do you mean by that? (If s/he uses expressions like steeper, higher, too uphill etc.)

If these two paths are not at same level of difficulty, what is the difference between the two?

How do you understand ?

Is the motorcycle charge only depended to the output height (or horizontal length)?

Can this motorcycle climb at the same height on different roads??

If you are going higher at the end of a road, is that road always steeper? Can you explain it by drawing?

What is the relationship between horizontal length and slope? Can you tell by drawing, or through examples?

Does the slope change depending on what? How?

2nd Clinical Interview Questions

The foothill seen in the picture will be made to provide access to the villas. At the entrance of each road there is a signboard  **showing the slope of the road. What do you need when you prepare the signboard?**

What do you need to be able to calculate the slope?

Could you show me the lengths you want with a pencil on paper?

How do you use those lengths you want now?

(If s/he claims the highest gives the slope) What will be written on signboard then?

How do you calculate the slope value of any kind of road, stairs, roof, ramp?

If this way starts a little higher up and signboard here (at the different location or point on the same linear road), is the value written on the signboard changed? Why?

Is there anything that does not change as the length of the path changes?

You say that the slope (or steepness) remains constant (if s/he can!). What makes the slope remain constant?

What lengths does the slope remain invariance?

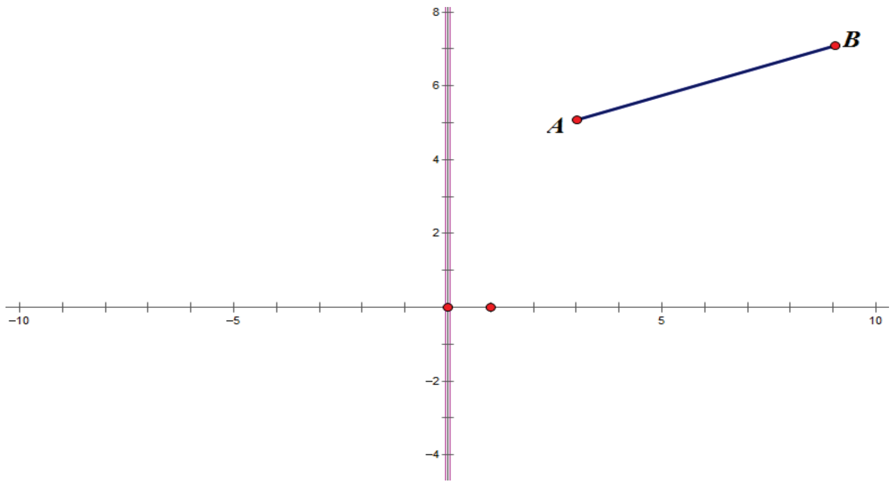
What relationship between these two ensures that the slope remains constant?

A mountain with a gradient of 3/10 is seen. Can you find the height of this mountain according to the length of the road which is parallel to this mountain and is 1000 m long? How?

Can you just calculate the slope of this part (by showing a piece on the hypotenuse of the right triangle model created by him/her on the mountain line that looks linear with the pen)?

How do you know? Why do you think so? How can you explain this?

3rd Clinical Interview Questions



Can you calculate the slope of the AB line you see?

Possible answers:

• By dividing the height into horizontal distances. **(So how do you find the height and the horizontal distance?)**

• I find it using the formula y_2-y_1 / x_2-x_1 **(Do you know where this formula comes from?)**

If s/he finds the rise over run (vertical distance/ horizontal distance),

Can you show the height and the horizontal distance by drawing the points?

How can you find the length of your height?

How do you find the length of the horizontal distance?

How easy is it to be in this coordinate plane?

Let me give you only two points, of course, I will give you the coordinates of this point, can you find the slope of the line that passes through those two points? (3, 5) and (7, 9) are given.

Possible answers:

• I will find two points on a coordinate plane. I combine the two to calculate the slope by rise over run.

Can you use these two points to calculate the slope without drawing a straight line?

Do you know a short way to calculate the slope using only two points?

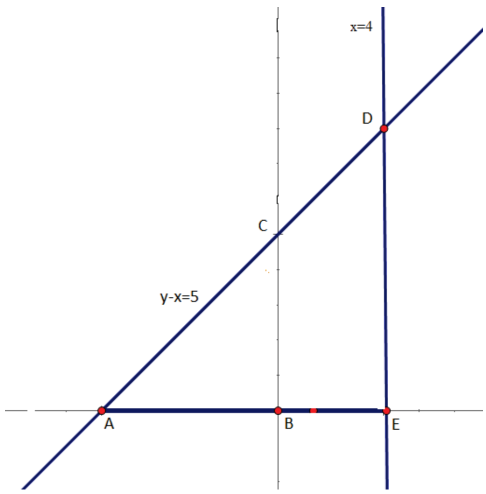
For example, can you find the slope of the line passing through points (685,350) and (385,850)?

Do you know a more practical way, as if it were going to be a little tricky in the way you did?

• I can find the slope by y_2-y_1/x_2-x_1 formulae.

How did you get this formula?

Can you calculate the slope without using this formula?



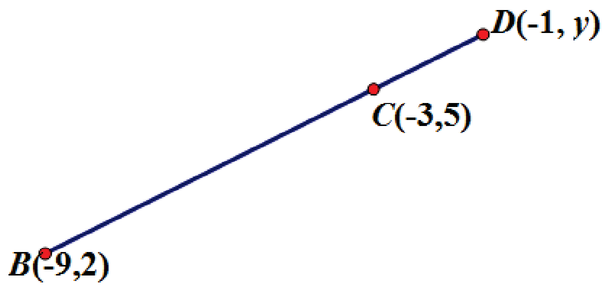
Calculate how many times the area of the ADE triangle, the area of the triangle ABC, according to data given in the coordinate plane.

Why did you use the slope in this question?

Or is there a relationship with the slope?

Does the slope help you?

How do you know that slope will not change in these points? Can you explain?



3 points are given on this line. Can you find the y coordinate of point D?

Why are you calculating the slope in this problem?

What is the relationship between the slope and this problem?

Why do you take these two points when calculating the slope? Why did not you take the others?

Why does the slope remain the same for any two points you get? How do you know it will be invariance?

Could you calculate the slope of a line if I gave you an equation of it? 5x + 4y-40 =0.

Can you calculate from different paths?

Why are you using this solution path?