Available online at www.sciencedirect.com
SCIENCE DIRECT®
Linear Algebra and its Applications 414 (2006) 547–559
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On the stability of a convex set of matrices

Vakif Dzhafarov *, Taner Büyükköroğlu

Department of Mathematics, Faculty of Science, Anadolu University, Eskisehir 26470, Turkey

Received 26 March 2005; accepted 31 October 2005
Available online 20 December 2005
Submitted by R.A. Brualdi

Abstract

In this paper we give an alternative proof of the constant inertia theorem for convex compact sets of complex matrices. It is shown that the companion matrix whose non-trivial column is negative satisfies the directional Lyapunov condition (inclusion) for real multiplier vectors. An example of a real matrix polytope that satisfies the directional Lyapunov condition for real multiplier vectors and which has nonconstant inertia is given. A new stability criterion for convex compact sets of real Z-matrices is given. This criterion uses only real vectors and positive definite diagonal matrices.

AMS classification: 15A24; 91A80; 93D30

Keywords: Stability; Constant inertia; Minimax theorem; Companion matrix

1. Introduction

Hurwitz stability and other related types of matrix stability play an important role in applications. Stability problems of the sets of matrices have been extensively studied in many works, see e.g. [1,3–19].

It is known that stability of $\mathscr{A}$, a compact set of complex matrices, may be characterized through a common solution of directional Lyapunov inclusions. Here we give an alternative proof this result. Then, we consider the case where $\mathscr{A}$ is real and explore under what conditions the inquired solution to the directional Lyapunov inclusions, may be confined to be real as well. Through an example it is demonstrated that in general this solution is complex. For the single

* Corresponding author.
E-mail addresses: vcaferov@anadolu.edu.tr (V. Dzhafarov), tbuyukkoroglu@anadolu.edu.tr (T. Büyükköroğlu).

0024-3795/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2005.10.044

companion matrix A whose non-trivial column is negative it is shown that the directional Lyapunov inclusion for real multiplier vectors is satisfied irrespective of the inertia of A. However, it is shown that the directional Lyapunov inclusions may be confined to be real whenever the set $\mathscr{A}$ consist of Z-matrices, where the solution may be further confined to be diagonal.

Let $\mathbb{C}^n$ ($\mathbb{R}^n$) be the set of complex (real) n vectors, $\mathbb{C}^{n\times n}$ ($\mathbb{R}^{n\times n}$) be the set of $n \times n$ complex (real) matrices. Let $\mathscr{H}$ denote the set of $n \times n$ complex Hermitian matrices, and $\mathscr{P}$ its subset of positive definite matrices.

Let $A \in \mathbb{C}^{n\times n}$. If A has ν (π) eigenvalues with negative (positive) real part and δ eigenvalues with zero real part then the triple (ν, δ, π) is called the inertia of A. In this definition algebraic multiplicities are taken into account. Therefore $\nu + \delta + \pi = n$. The matrix A is called Hurwitz stable (positive stable) if $\nu = n$ ($\pi = n$). The inertia is said to be regular if $\delta = 0$. A is Hurwitz stable if and only if $-A$ is positive stable. A^* will denote the complex conjugate transpose of a matrix A.

In the following we give a well-known theorem from [13] for stability and regularity of inertia.

Theorem 1.1 [13, p. 107]. *Let $A \in \mathbb{C}^{n\times n}$. Then A has regular inertia* (*is positive stable*) *if and only if for every nonzero $z \in \mathbb{C}^n$ there exists Hermitian* (*positive definite*) *matrix H such that*

$$\operatorname{Re} z^* HAz > 0.$$

For $A \in \mathbb{C}^{n\times n}$ and nonzero $z \in \mathbb{C}^n$ define

$$\mathscr{H}(A, z) = \{H \in \mathscr{H} : \operatorname{Re} z^* HAz > 0\}.$$

If $\mathscr{E}$ is a set of matrices, i.e. if $\mathscr{E} \subset \mathbb{C}^{n\times n}$, then define

$$\mathscr{H}(\mathscr{E}, z) = \bigcap_{A\in\mathscr{E}} \mathscr{H}(A, z). \tag{1.1}$$

As follows from (1.1), $H \in \mathscr{H}(\mathscr{E}, z)$ if and only if $H \in \mathscr{H}$ and

$$\operatorname{Re} z^* HAz > 0 \quad \text{for all } A \in \mathscr{E}.$$

Let $\mathscr{E}$ be a compact subset in $\mathbb{C}^{n\times n}$ and $\mathscr{A}$ be the convex hull of $\mathscr{E}$: $\mathscr{A} = \operatorname{conv}(\mathscr{E})$. From convex analysis it follows that $\mathscr{A}$ is also compact. (In finite dimensional spaces, if the set is compact then its convex hull is also compact, converse is not true in general.) The family $\mathscr{A}$ is called with regular inertia (Hurwitz stable) if all matrices in $\mathscr{A}$ are with regular inertia (Hurwitz stable).

The stability of $\mathscr{A}$ for the case where $\mathscr{E} = \{A, B\}$ (i.e. $\mathscr{E}$ a doubleton) was established in [14]. In [6] this was extended to any compact $\mathscr{E}$; and in turn generalized in [8] to the following on regular inertia.

Theorem 1.2 [8]. *Let $\mathscr{E} \subset \mathbb{C}^{n\times n}$ be a compact set of matrices and let $\mathscr{A}$ be the convex hull of $\mathscr{E}$. Then the following are equivalent*:

(i) *All matrices in $\mathscr{A}$ have the same regular inertia.*
(ii) *$\mathscr{H}(\mathscr{A}, z) \neq \emptyset$ for every nonzero $z \in \mathbb{C}^n$.*
(iii) *$\mathscr{H}(\mathscr{E}, z) \neq \emptyset$ for every nonzero $z \in \mathbb{C}^n$.*

From continuity property of the roots of polynomials it follows that in the context of a convex set of matrices "regular inertia" and "constant regular inertia" are equivalent. (This was pointed out in [9].)

If complex vector z is real, it will be denoted by x. For $\mathscr{E} \subset \mathbb{R}^{n\times n}$ define

$$\mathscr{S}(\mathscr{E}, x) = \mathscr{H}(\mathscr{E}, x) \cap \mathbb{R}^{n\times n}. \tag{1.2}$$

As follows from (1.2), for $\mathscr{E} \subset \mathbb{R}^{n\times n}$ and nonzero $x \in \mathbb{R}^n$ the matrix H belongs to $\mathscr{S}(\mathscr{E}, x)$ if and only if H is real, symmetric and

$$x^{\mathrm{T}} H A x > 0$$

for all $A \in \mathscr{E}$, where the symbol T denotes the transpose.

As pointed out above this manuscript mainly addresses three points:

(1) An alternative proof of Theorem 1.2 based on the minimax theorem.
(2) Investigating the specialization of Theorems 1.1 and 1.2 to the case where $\mathscr{E} \subset \mathbb{R}^{n\times n}$. Specifically, under what conditions the search in $\mathscr{H}(\mathscr{E}, z)$ may be confined to $\mathscr{S}(\mathscr{E}, x)$. First, when $\mathscr{E}$ consists of a single companion matrix A whose non-trivial column is negative, it is shown (Proposition 3.1) that the condition on $\mathscr{S}(A, x)$ is satisfied irrespective of the inertia of A. Thus in Theorem 1.1 the search over the larger set $\mathscr{H}(A, z)$ in general cannot be avoided. It is then illustrated through Example 4.1 that a similar conclusion holds for Theorem 1.2.
(3) Exploring the case where $\mathscr{E}$ is comprised of Z-matrices. Here, it is shown (Theorem 5.3) that the set $\mathscr{H}(\mathscr{E}, z)$ may be confined not only to $\mathscr{S}(\mathscr{E}, x)$, but in addition all matrices in $\mathscr{S}$ may be diagonal.

At the end of the paper we give illustrative examples, where the computational complexity is considered.

2. Application of the minimax theorem

In this section we show that constant inertia theorem (Theorem 1.2) follows from classical minimax theorem of the game theory. We begin with formulation of the minimax theorem in a form which we need (see, for example [2, Subsection 8.3], [20, p. 393]).

Theorem 2.1 (Minimax theorem). *Let X and Y be normed spaces, E a compact convex subset of X and F a convex subset of Y. Let the function $f : E \times F \to \mathbb{R}$ satisfy the following conditions*:

(1) *For all $y \in F$ the function $x \to f(x, y)$ is convex and continuous*,
(2) *For all $x \in E$ the function $y \to f(x, y)$ is concave*.

Then

$$\inf_{x\in E} \sup_{y\in F} f(x, y) = \sup_{y\in F} \inf_{x\in E} f(x, y).$$

In the game theory this theorem is called a nonsymmetric minimax theorem.

Proof of Theorem 1.2. The implication (ii) $\Rightarrow$ (i) follows from Theorem 1.1. To prove the implication (i) $\Rightarrow$ (ii) choose arbitrary $A_0 \in \mathscr{A}$ and fix arbitrary nonzero vector $z \in \mathbb{C}^n$. Since A_0 has regular inertia then by Theorem 1.1 there exists $H_0 \in \mathscr{H}$ such that

$$\operatorname{Re} z^* H_0 A_0 z = \alpha > 0 \tag{2.1}$$

(α depends on z and A_0).

From (2.1) for real positive scalar λ we have

$$\operatorname{Re} z^*(\lambda H_0)A_0 z = \lambda\alpha. \tag{2.2}$$

From (2.2) we have

$$\sup_{H\in\mathscr{H}} \operatorname{Re} z^* H A_0 z \geqslant \sup_{\lambda>0} \operatorname{Re} z^*(\lambda H_0)A_0 z = \sup_{\lambda>0} \lambda\alpha = +\infty. \tag{2.3}$$

Therefore the left-hand side of (2.3) is infinity and, as $A_0 \in \mathscr{A}$ is arbitrary, we get

$$\inf_{A\in\mathscr{A}} \sup_{H\in\mathscr{H}} \operatorname{Re} z^* H A z = +\infty. \tag{2.4}$$

Let the function $f(A, H) : \mathscr{A} \times \mathscr{H} \to \mathbb{R}$ be defined as

$$f(A, H) = \operatorname{Re} z^* H A z.$$

Then f is continuous and linear with respect to A and H. By minimax theorem (where $E = \mathscr{A}$, $F = \mathscr{H}$) and (2.4) we have

$$\sup_{H\in\mathscr{H}} \inf_{A\in\mathscr{A}} \operatorname{Re} z^* H A z = +\infty. \tag{2.5}$$

From (2.5) it follows that there exists $H = H(z) \in \mathscr{H}$ such that

$$\inf_{A\in\mathscr{A}} \operatorname{Re} z^* H A z > 1,$$

or for all $A \in \mathscr{A}$

$$\operatorname{Re} z^* H A z > 1.$$

Therefore (ii) is satisfied. □

An alternative proof of the stability criterion in [6,14] may be obtained if Theorem 1.1 is combined with the above proof of Theorem 1.2, where the set $\mathscr{H}$ is substituted by $\mathscr{P}$.

It should be noted that another alternative proof of Theorem 1.2 based on the geometry of convex sets is given in [17].

3. Companion matrices

It is well known from stability theory of polynomials that a companion matrix with negative non-trivial column may be Hurwitz stable or may have mixed inertia or non-regular inertia, see e.g. [3,5]. Here it is shown that such a matrix satisfies the directional Lyapunov condition for real multiplier vectors, irrespective of the inertia. Thus in Theorem 1.1 the search over the larger set $\mathscr{H}(A, z)$ in general cannot be avoided. A similar proposition can be proven for the compact family of companion matrices, but we omit this generalization.

Let

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \tag{3.1}$$

be a companion matrix. Characteristic polynomial $\det(sI - A)$ of the matrix (3.1) is the monic polynomial ("det" denotes determinant)

$$p(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0.$$

The necessary but not sufficient condition for Hurwitz stability of the matrix (3.1) is that all of $a_0, a_1, \ldots, a_{n-1}$ have positive sign. For $n = 1, 2$ this condition is sufficient as well. It can be easily shown that characteristic polynomial of the convex combination of the companion matrices is equal to the convex combination of the corresponding monic polynomials. It is well-known that the stability of two companion matrices does not guarantee the stability of all convex combinations.

Proposition 3.1. *Let A be a companion matrices with negative non-trivial column. Then for every nonzero $x \in \mathbb{R}^n$ there exists a positive definite real matrix $P = P(x)$ such that*

$$x^{\mathrm{T}} PAx < 0. \tag{3.2}$$

To prove Proposition 3.1 we need two lemmas.

Lemma 3.2. *For every nonzero $(x_1, x_2, \ldots, x_{n-1}, 0)^{\mathrm{T}}$ there exists a real positive definite matrix $P = P(x)$ such that*

$$(x_1, x_2, \ldots, x_{n-1}, 0).P.(0, x_1, x_2, \ldots, x_{n-1})^{\mathrm{T}} < 0. \tag{3.3}$$

Proof. Consider the Hurwitz stable polynomial $(s + 1)^n$ and let

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\delta_0 \\ 1 & 0 & \cdots & 0 & -\delta_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\delta_{n-1} \end{pmatrix}$$

be its companion matrix. The matrix L is Hurwitz stable and by the real version of the Lyapunov theorem [13, p. 96] there exists positive definite $P \in \mathbb{R}^{n\times n}$ such that $L^{\mathrm{T}}P + PL$ is negative definite, or equivalently

$$x^{\mathrm{T}} PLx < 0 \tag{3.4}$$

for all nonzero $x^{\mathrm{T}} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. On the other hand

$$Lx = (-\delta_0 x_n, x_1 - \delta_1 x_n, x_2 - \delta_2 x_n, \ldots, x_{n-1} - \delta_{n-1} x_n)^{\mathrm{T}}. \tag{3.5}$$

If we set $x_n = 0$ in (3.4) then from (3.4), (3.5) we obtain (3.3). □

Lemma 3.3. *Consider*

$$F_0 = -a_0 x_1 x_n,\ F_1 = x_2(x_1 - a_1 x_n),\ F_2 = x_3(x_2 - a_2 x_n), \ldots, F_{n-1} = x_n(x_{n-1} - a_{n-1} x_n).$$

Here a_i are arbitrary fixed positive numbers ($i = 0, 1, \ldots, n - 1$). Then for all $(x_1, x_2, \ldots, x_n)$ with $x_n \neq 0$ there exists an index $m \in \{0, 1, \ldots, n - 1\}$ such that

$$F_m < 0.$$

Proof. Without loss of generality assume that $x_n > 0$. If

$$x_1 \leqslant 0, x_2 \leqslant 0, \ldots, x_{n-1} \leqslant 0$$

then $F_{n-1} < 0$. If there exists k such that

$$x_1 \leqslant 0, x_2 \leqslant 0, \ldots, x_k \leqslant 0, x_{k+1} > 0$$

then $F_k < 0$. If $x_1 > 0$ then $F_0 < 0$. □

Proof of Proposition 3.1. Let arbitrary nonzero $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$ be given. If $x_n = 0$ then by Lemma 3.2 there exists positive definite $P \in \mathbb{R}^{n\times n}$ such that

$$x^{\mathrm{T}} P A x = (x_1, x_2, \ldots, x_{n-1}, 0) \cdot P \cdot (0, x_1, x_2, \ldots, x_{n-1})^{\mathrm{T}} < 0.$$

If $x_n \neq 0$ then for the diagonal matrix $D = \operatorname{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n\times n}$ we have

$$x^{\mathrm{T}} D A x = \lambda_0 F_0 + \lambda_1 F_1 + \cdots + \lambda_{n-1} F_{n-1}$$

where F_i is defined as in Lemma 3.3. Then by Lemma 3.3 there exists an index m such that $F_m < 0$. Then by choosing λ_i sufficiently small positive for $i \neq m$ and sufficiently large for $i = m$ we can guarantee the inequality

$$x^{\mathrm{T}} D A x < 0.$$

Summarizing, for any nonzero $x \in \mathbb{R}^n$ we can choose positive definite $P = P(x)$ such that

$$x^{\mathrm{T}} P A x < 0. \qquad \square$$

Concluding this section, we point out that for the companion matrix A an effective analytic procedure for solving the Lyapunov equation

$$A^{\mathrm{T}} P + P A = -Q$$

is given in [4] where P and Q are symmetric matrices.

4. An illustrative example

In the previous section we illustrated the fact that in Theorem 1.1 even when the matrix A is real, one cannot confine the matrices H and the vectors z to be real. In fact, as pointed out in subsection 9.1 in [9], by doubling the dimension one can re-state Theorems 1.1 and 1.2 in terms of real vectors and matrices. Indeed, every matrix A and every Hermitian matrix H can be written as $A = A_R + \mathrm{j}A_I$, $H = S + \mathrm{j}T$, where S is symmetric and T is skew symmetric. Similarly the vector $z \in \mathbb{C}^n$ can be written as $z = x + \mathrm{j}y$ with $x, y \in \mathbb{R}^n$. Define the following real matrices and vector:

$$\hat{A} := \begin{pmatrix} A_R & A_I \\ -A_I & A_R \end{pmatrix}, \quad \hat{S} := \begin{pmatrix} S & T \\ -T & S \end{pmatrix}, \quad \hat{x} := \begin{pmatrix} x \\ -y \end{pmatrix}.$$

It is easy to verify that

$$\operatorname{Re} z^* H A z = \hat{x}^{\mathrm{T}} \hat{S} \hat{A} \hat{x}. \tag{4.1}$$

Theorem 1.2 (and Theorem 1.1) may be easily formulated in this framework. Indeed, define

$$\hat{\mathscr{E}} = \left\{ \hat{A} \in \mathbb{R}^{2n\times 2n} : A \in \mathscr{E} \right\}.$$

Then the statement (iii) of Theorem 1.2 is equivalent to the following: For every nonzero $\hat{x} \in \mathbb{R}^{2n}$ there exists $\hat{S} = \hat{S}(\hat{x})$ such that

$$\hat{x}^{\mathrm{T}} \hat{S} \hat{A} \hat{x} > 0 \tag{4.2}$$

for all $\hat{A} \in \hat{\mathscr{E}}$.

Now from (4.1) it follows that for $A \in \mathbb{R}^{n\times n}$, namely where $A = A_R$,

$$\hat{x}^{\mathrm{T}} \hat{S} \hat{A} \hat{x}|_{A_I=0} = x^{\mathrm{T}} S A_R x + y^{\mathrm{T}} T A_R x - x^{\mathrm{T}} T A_R y + y^{\mathrm{T}} S A_R y.$$

Theorem 1.2 requires the above quantity to be positive for all (up to possible positive scaling) nonzero $x, y \in \mathbb{R}^n$. In Example 4.1, it is illustrated that one cannot restrict the discussion for the case where $y = 0$.

Example 4.1. Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues of A are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{j}$, and A is Hurwitz stable.

At first we will show that for every nonzero $x \in \mathbb{R}^2$ there exists a positive definite $P = P(x) \in \mathbb{R}^{2\times 2}$ such that

$$x^{\mathrm{T}} P A x > 0. \tag{4.3}$$

Without loss of generality all 2×2 positive definite matrices may be normalized to have trace 2, i.e.

$$P = \begin{pmatrix} 1+\alpha & \beta \\ \beta & 1-\alpha \end{pmatrix}$$

and this is parameterized by the unit disc in $\mathbb{R}^2$, namely $\alpha^2 + \beta^2 < 1$, see e.g. [6,Section 5], [9, Section 5]. Then for $x^{\mathrm{T}} = (1, r)$,

$$x^{\mathrm{T}} P A x = r^2(\alpha - \beta - 1) + r(-2\alpha - \beta) + \beta$$

and, one can take the following positive semi-definite matrices described by (α, β) pairs: $(0, -1)$ for $r > \frac{5}{4}$, $(-\frac{4}{5}, -\frac{3}{5})$ for $\frac{5}{12} < r \leqslant \frac{5}{4}$, $(0, 1)$ for $-\frac{1}{2} \leqslant r \leqslant \frac{5}{12}$, $(1, 0)$ for $r < -\frac{1}{2}$. Up to an ϵ-perturbation, so that each of the P matrices is positive definite, the first setup is complete.

Now let $\mathscr{E} = \{A, I\}$. Clearly every positive definite matrix $P = P(x)$ satisfying (4.3) also guarantees that

$$x^{\mathrm{T}} P I x = x^{\mathrm{T}} P x > 0.$$

Thus $\mathscr{S}(\mathscr{E}, x) \neq \emptyset$ for all $0 \neq x \in \mathbb{R}^2$. However, since A is Hurwitz stable and I is positively stable, $\mathscr{A}$ does not have constant regular inertia. Indeed,

$$C = \frac{2}{3}A + \frac{1}{3}I = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$$

has pure imaginary eigenvalues. This shows that for the subset $\mathscr{E} \subset \mathbb{R}^{n\times n}$ in Theorem 1.2 the search in $\mathscr{H}(\mathscr{E}, z)$ cannot be substituted by $\mathscr{S}(\mathscr{E}, x)$. (The possibility of such substitution for real matrices was stated in Theorem 7.2 from [9].) □

5. *Z*-matrices

In this section, using the minimax approach from Section 2 we give a new robust stability criterion for subsets of Z-matrices. In this criterion the vector $\hat{x}$ from (4.1) will vary over $\mathbb{R}^n$, and directional Lyapunov factors are diagonal matrices.

Definition 5.1 [13, p. 113]. A real $n \times n$ matrix $A = (a_{ij})$ is said to be Z-matrix if $a_{ij} \leqslant 0$ for all $i \neq j$.

Denote the set of real $n \times n$, Z-matrices by $\mathscr{Z}$, and denote by $\mathscr{D}$ the set of $n \times n$ real positive diagonal matrices, i.e.

$$\mathscr{D} = \{\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i > 0\ (i = 1, 2, \dots, n)\}.$$

The following theorem can be found in [13].

Theorem 5.2 [13, p. 114]. *Let the real matrix $A \in \mathscr{Z}$ be given. Then the following are equivalent.*

(1) *A is positive stable.*
(2) *For each nonzero $x \in \mathbb{R}^n$ there exists $D = D(x) \in \mathscr{D}$ such that $x^{\mathrm{T}} DAx > 0$.*

Convex combination of two positive stable Z-matrices needs not be positive stable.
For $\mathscr{E} \subset \mathbb{R}^{n\times n}$ and nonzero $x \in \mathbb{R}^n$ using (1.2) define

$$\mathscr{D}(\mathscr{E}, x) := \mathscr{S}(\mathscr{E}, x) \cap \mathscr{D}.$$

Now we formulate a new stability criterion for compact convex subsets of $\mathscr{Z}$.

Theorem 5.3. *Let $\mathscr{E} \subset \mathscr{Z}$ be compact and $\mathscr{A} = conv(\mathscr{E})$. Then the following are equivalent.*

(i) *All matrices in $\mathscr{A}$ are positively stable.*
(ii) *$\mathscr{D}(\mathscr{A}, x) \neq \emptyset$ for all nonzero $x \in \mathbb{R}^n$.*
(iii) *$\mathscr{D}(\mathscr{E}, x) \neq \emptyset$ for all nonzero $x \in \mathbb{R}^n$.*

Proof. By definition if $\mathscr{E} \subset \mathscr{Z}$ then $\mathscr{A} = \text{conv}(\mathscr{E}) \subset \mathscr{Z}$. Since $\mathscr{D}(\mathscr{A}, x) = \mathscr{D}(\mathscr{E}, x)$, (ii) and (iii) are equivalent.

The implication (ii) $\Rightarrow$ (i) follows from Theorem 5.2. To prove the implication (i) $\Rightarrow$ (ii) choose arbitrary $A_0 \in \mathscr{A}$ and fix arbitrary nonzero $x \in \mathbb{R}^n$. Since A_0 is positive stable by Theorem 5.2 there exists $D_0 \in \mathscr{D}$ such that

$$x^{\mathrm{T}} D_0 A_0 x = \alpha > 0$$

(α depends on x and A_0).

For $\lambda > 0$ we have

$$x^{\mathrm{T}}(\lambda D_0) A_0 x = \lambda\alpha.$$

Therefore

$$\sup_{D\in\mathscr{D}} x^{\mathrm{T}} D A_0 x \geqslant \sup_{\lambda>0} x^{\mathrm{T}}(\lambda D_0) A_0 x = \sup_{\lambda>0} \lambda\alpha = +\infty.$$

As $A_0 \in \mathscr{A}$ is arbitrary we have

$$\inf_{A\in\mathscr{A}} \sup_{D\in\mathscr{D}} x^{\mathrm{T}} DAx = +\infty. \tag{5.1}$$

Let the continuous function $f : \mathscr{A} \times \mathscr{D} \to R$ be defined as

$$f(A, D) = x^{\mathrm{T}} DAx$$

which is linear with respect to A and D. Then by minimax theorem and (5.1) we obtain

$$\sup_{D\in\mathscr{D}} \inf_{A\in\mathscr{A}} x^{\mathrm{T}} DAx = +\infty.$$

This equality as in the proof of Theorem 1.2 implies that for a given x there exists $D \in \mathscr{D}$ such that

$$x^{\mathrm{T}} DAx > 1$$

for all $A \in \mathscr{A}$, therefore $\mathscr{D}(\mathscr{A}, x) \neq \emptyset$. □

Thus, for Z-matrices we obtained a twofold improvement of the condition (iii) of Theorem 1.2: The common directional Lyapunov factor may be confined to positive diagonal matrices and the multiplier vector may be restricted to the unit sphere in $\mathbb{R}^n$ rather than $\mathbb{C}^n$.

Note that the existence of a common diagonal solution to a Lyapunov equation associated with a set of upper triangular nonsingular complex matrices was studied in [7, Section 5]. There it was proven under mild restrictions that if these matrices share the same sign of the real part of the diagonals, then this family has a common Lyapunov solution in a diagonal form. In [6,17] it was shown that if $\mathscr{A}$ is a convex set of upper triangular positive stable complex matrices, then directional Lyapunov inclusion has common diagonal solutions.

Corollary 5.4. *Let the set $\mathscr{E} \subset \mathbb{R}^{n\times n}$ be compact and for every $A \in \mathscr{E}$ all nondiagonal elements of A be nonnegative (i.e., $-\mathscr{E} \subset \mathscr{Z}$). Let $\mathscr{A} = \operatorname{conv}(\mathscr{E})$. Then $\mathscr{A}$ is robustly Hurwitz stable if and only if for every nonzero $x \in \mathbb{R}^n$ there exists $D = D(x) \in \mathscr{D}$ such that for all $A \in \mathscr{E}$*

$$x^{\mathrm{T}} DAx < 0.$$

If all principal minors of a matrix $A \in \mathbb{R}^{n\times n}$ are positive then A is said to be a P-matrix. For a 2×2 matrix we can prove the following

Proposition 5.5. *For $A \in \mathbb{R}^{2\times 2}$ the following are equivalent*:

(i) *$-A$ is a P-matrix.*
(ii) *There exists a positive diagonal matrix D so that $-(DA + A^{\mathrm{T}}D)$ is positive definite.*
(iii) *For every nonzero $x \in \mathbb{R}^2$ there exists a positive diagonal matrix $D = D(x)$ so that $x^{\mathrm{T}}DAx < 0$.*

In particular this implies that A is Hurwitz stable.

Proof. The equivalence of (i) and (ii) is due to G.W. Cross, see e.g. [16, Fact 2.8.1]. Trivially, (ii) implies (iii). Let for the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (iii) be satisfied and $x = (x_1, x_2)^{\mathrm{T}}$, $D = \operatorname{diag}(\lambda_1, \lambda_2)$. Then by (iii) for every nonzero $x \in \mathbb{R}^2$ there exist $\lambda_1 > 0$, $\lambda_2 > 0$ such that

$$x^{\mathrm{T}} DAx = \lambda_1\left(ax_1^2 + bx_1x_2\right) + \lambda_2\left(cx_1x_2 + dx_2^2\right) < 0.$$

For the vectors $x = (1, 0)^{\mathrm{T}}$, $x = (0, 1)^{\mathrm{T}}$ and $x = (b, -a)^{\mathrm{T}}$ this inequality gives $a < 0$, $d < 0$ and $ad - bc > 0$. Consequently (iii) implies (i). □

For the compact convex set of 2×2 dimensional P-matrices we have the following.

Proposition 5.6. *Let the set $\mathscr{E} \subset \mathbb{R}^{2\times 2}$ be a compact set of P-matrices and $\mathscr{A} = \operatorname{conv}(\mathscr{E})$. Then the following are equivalent*:

(i) *All matrices in* $\mathscr{A}$ *are positively stable.*
(ii) $\mathscr{H}(\mathscr{E}, z) \neq \emptyset$ *for all nonzero* $z \in \mathbb{C}^2$.
(iii) $\mathscr{S}(\mathscr{E}, x) \neq \emptyset$ *for all nonzero* $x \in \mathbb{R}^2$.
(iv) $\mathscr{D}(\mathscr{E}, x) \neq \emptyset$ *for all nonzero* $x \in \mathbb{R}^2$.

Proof. Since any 2×2 dimensional P-matrix is positively stable then the equivalence of (i) and (ii) follows from Theorem 1.2. Trivially (iv) implies (iii). If (iv) is satisfied then $\mathscr{D}(A, x) \neq \emptyset$ for every $A \in \mathscr{A}$ and nonzero $x \in \mathbb{R}^2$. Then by [13, Theorem 2.5.6] every matrix in $\mathscr{A}$ is a 2×2 dimensional P-matrix and consequently $\mathscr{A}$ is positively stable and the implication (iv) $\to$ (i) is true.

The implication (i) $\to$ (iv) follows from [13, Theorem 2.5.6] and the minimax theorem (Theorem 2.1). Indeed, let (i) be true. Then every matrix in $\mathscr{A}$ is a P-matrix (diagonal entries are positive as convex combination of positive numbers; from positive stability it follows that determinant is also positive). Choose arbitrary $A_0 \in \mathscr{A}$ and nonzero $x \in \mathbb{R}^2$. Then by [13, Theorem 2.5.6] there exists $D_0 \in \mathscr{D}$ such that

$$x^{\mathrm{T}} D_0 A_0 x > 0.$$

From this we obtain $\mathscr{D}(\mathscr{A}, x) \neq \emptyset$ exactly as in the proof of Theorem 5.3 and the implication (i) $\to$ (iv) is established.

Finally, the implication (iii) $\to$ (i) follows from continuity. Indeed, by contrary, assume that (iii) is true, but (i) is not. Assume the matrix $A \in \mathscr{A}$ is not positively stable. Since the family $\mathscr{A}$ contains also a positively stable matrix B (recall that every matrix from $\mathscr{E}$ is a 2×2 dimensional P-matrix and consequently is positively stable) then

$$\det(A) \leqslant 0, \quad \det(B) > 0$$

From this by continuity it follows that there exists $C \in \mathscr{A}$ such that $\det(C) = 0$. Then there exists a nonzero $x_* \in \mathbb{R}^2$ such that $Cx_* = 0$. In this case (iii) will be violated for this x_*. Consequently the implication (iii) $\to$ (i) is true and the proof is complete. □

The obtained results has some computational advantages. We have restricted the search x in the unit sphere in $\mathbb{R}^n$ rather $\mathbb{C}^n$, and common directional Lyapunov factor is confined to diagonal matrices. According to [8] we are searching for a finite covering $\mathscr{X}_1, \mathscr{X}_2, \ldots, \mathscr{X}_m$ of the unit sphere in $\mathbb{R}^n$ and matrices $D_1, D_2, \ldots, D_m \in \mathscr{D}$ such that $D_i \in \mathscr{D}(\mathscr{E}, \mathscr{X}_i)$ $(i = 1, 2, \ldots, m)$. Whenever a minimal m exists, it may serve as an indication for the computational complexity involved.

It is well known that [6,8] in searching for common $D_1, D_2, \ldots$ we have to solve the constrained optimization problems. To this end define the norm in $\mathbb{R}^n$ by $\|x\| = \max_i |x_i|$. Unit sphere in this norm is the set

$$\big\{(x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n : -1 \leqslant x_i \leqslant 1 \ (i = 1, 2, \ldots, n) \\ \text{and there exists } k \text{ such that } x_k = \pm 1\big\}$$

and it is naturally partitioned into $2n$ boxes from $\mathbb{R}^{n-1}$. By symmetry one can restrict the search x in the faces $x_i = 1$ $(i = 1, 2, \ldots, n)$ and then attempt to find the common matrices $D_1, D_2, \ldots$ for each face separately.

(By doubling the dimension the same partition of the unit sphere can be considered in the complex case also, see (4.2)).

Let 2×2 real matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.2}$$

be given and $a > 0$, $d > 0$, $ad - bc > 0$, (i.e. the matrix A is P-matrix). Let $x^{\mathrm{T}} = (x_1, x_2)$ and $D = \operatorname{diag}(\lambda, 1)$ where λ is positive parameter. Then

$$\begin{aligned} x^{\mathrm{T}} DAx &= a\lambda x_1^2 + (b\lambda + c)x_1x_2 + dx_2^2 \\ &= \left(\frac{b\lambda + c}{2\sqrt{d}}x_1 + \sqrt{d}x_2\right)^2 + x_1^2\left(a\lambda - \frac{(b\lambda + c)^2}{4d}\right) \end{aligned}$$

The expression in the second bracket is positive if and only if

$$\lambda \in (\lambda_1, \lambda_2), \tag{5.3}$$

where

$$\begin{aligned} \lambda_1 &= \frac{2ad - bc - 2\sqrt{a^2d^2 - abcd}}{b^2}, \\ \lambda_2 &= \frac{2ad - bc + 2\sqrt{a^2d^2 - abcd}}{b^2}. \end{aligned}$$

(If $b = 0$ then $\lambda_1 = c^2/(4ad)$, $\lambda_2 = +\infty$.) Therefore for the matrix (5.2) with $a > 0$, $d > 0$, $ad - bc > 0$ there exists interval (λ_1, λ_2) such that:

$$\text{If } \lambda \in (\lambda_1, \lambda_2) \text{ then } x^{\mathrm{T}}DAx > 0 \text{ for all nonzero } x \in \mathbb{R}^2. \tag{5.4}$$

$$\text{If } \lambda \notin (\lambda_1, \lambda_2) \text{ then } x^{\mathrm{T}}DAx \leqslant 0 \text{ for some nonzero } x \in \mathbb{R}^2. \tag{5.5}$$

Example 5.7. Consider P-matrices

$$A_1 = \begin{pmatrix} 1 & 5 \\ 0.1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 34 & -1 \\ -21 & 2 \end{pmatrix}$$

and $\mathscr{A} = \operatorname{conv}(A_1, A_2)$. Since the product matrix (A_1A_2) has a real negative eigenvalue (in fact two) from [9, Lemma 6.1] it follows that $\bigcap_{x \in \mathbb{R}^n} \mathscr{S}(\mathscr{E}, x) = \emptyset$. Namely there is no single real symmetric matrix S, let alone diagonal, which satisfies condition (ii) in Proposition 5.5 simultaneously for all matrices is $\mathscr{A}$.

On the other hand the minimization problems for the subsets of the faces $x_1 = 1$ and $x_2 = 1$ give two diagonal matrices in common that guarantee for every matrix in $\mathscr{A}$ to be P-matrix. These matrices can be chosen as

$$\begin{aligned} &\operatorname{diag}(0.1, 1) \quad \text{for the subsets } \left\{(x_1, 1) : -1 \leqslant x_1 \leqslant \tfrac{1}{34}\right\}, \{(1, x_2) : -1 \leqslant x_2 \leqslant 0\}; \\ &\operatorname{diag}(2, 1) \quad \text{for the subsets } \left\{(x_1, 1) : \tfrac{1}{34} \leqslant x_1 \leqslant 1\right\}, \{(1, x_2) : 0 \leqslant x_2 \leqslant 1\}. \qquad \square \end{aligned}$$

Example 5.8. Consider the positively stable Z-matrices

$$A_1 = \begin{pmatrix} 3 & 0 & -2 \\ -2 & 1 & 0 \\ -1 & -0.5 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & 0 & 0 \\ -2 & 2 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$

and $\mathscr{A} = \operatorname{conv}(A_1, A_2)$. Let $x^{\mathrm{T}} = (x_1, x_2, x_3)^{\mathrm{T}}$, $D = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$.

For $x_1 = x_2 = 1$, $x_3 = 2$ the inequalities

$$x^{\mathrm{T}} DA_1 x > 0, \quad x^{\mathrm{T}} DA_2 x > 0$$

become

$$-\lambda_1 - \lambda_2 + 5\lambda_3 > 0, \quad 0.5\lambda_1 - 4\lambda_3 > 0$$

which give

$$-\lambda_2 - 3\lambda_3 > 0.$$

The last inequality contradicts the positivity of λ_2 and λ_3. Therefore by Theorem 5.3 the family $\mathscr{A}$ is not positive stable. $\square$

Example 5.9. Consider the positively stable Z-matrices

$$A_1 = \begin{pmatrix} 34 & -1 & -1 \\ -21 & 2 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -15 & -1 \\ -1 & 26 & 0 \\ -1 & -2 & 3 \end{pmatrix}.$$

and $\mathscr{A} = \mathrm{conv}(A_1, A_2)$. Let $x^{\mathrm{T}} = (x_1, x_2, x_3)^{\mathrm{T}}$, $D = \mathrm{diag}(\lambda_1, \lambda_2, 1)$. In this example the unique diagonal matrix that guarantees the positive stability of the family $\mathscr{A}$ does not exist. To see this consider the 2×2 leading principal submatrices of A_1 and A_2. If the unique diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, 1)$ exists then the 2×2 diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2)$ would be the common diagonal matrix for these submatrices. On the other hand it is impossible, since the intervals (5.3) calculated for these matrices have empty intersection. In the following we list the diagonal matrices D that guarantee the positivity of $x^{\mathrm{T}} DA_1 x$ and $x^{\mathrm{T}} DA_2 x$ in different subsets of the faces $x_1 = 1$, $x_2 = 1$ and $x_3 = 1$.

Face $x_1 = 1$:

$$\begin{aligned}
&\mathrm{diag}(1, 1, 1) && \text{for } \left\{(x_2, x_3) : -1 \leqslant x_2 \leqslant \tfrac{1}{24} \text{ or } 0.61 \leqslant x_2 \leqslant 1,\ -1 \leqslant x_3 \leqslant 1\right\},\\
&\mathrm{diag}(1, 9, 1) && \text{for } \left\{(x_2, x_3) : \tfrac{1}{24} \leqslant x_2 \leqslant \tfrac{1}{8},\ -1 \leqslant x_3 \leqslant 1\right\},\\
&\mathrm{diag}(1, 5, 1) && \text{for } \left\{(x_2, x_3) : \tfrac{1}{8} \leqslant x_2 \leqslant \tfrac{1}{4},\ -1 \leqslant x_3 \leqslant 1\right\},\\
&\mathrm{diag}(1, 2.4, 1) && \text{for } \left\{(x_2, x_3) : \tfrac{1}{4} \leqslant x_2 \leqslant 0.61,\ -1 \leqslant x_3 \leqslant 1\right\}.
\end{aligned}$$

Face $x_2 = 1$:

$$\begin{aligned}
&\mathrm{diag}(1, 0.22, 1) && \text{for } \left\{(x_1, x_3) : -1 \leqslant x_1 \leqslant \tfrac{1}{3}, -1 \leqslant x_3 \leqslant 1\right\},\\
&\mathrm{diag}(2, 0.9, 1) && \text{for } \left\{(x_1, x_3) : \tfrac{1}{3} \leqslant x_1 \leqslant \tfrac{1}{2}, -1 \leqslant x_3 \leqslant 1\right\},\\
&\mathrm{diag}(2, 1.3, 1) && \text{for } \left\{(x_1, x_3) : \tfrac{1}{2} \leqslant x_1 \leqslant 1, -1 \leqslant x_3 \leqslant 1\right\}.
\end{aligned}$$

Face $x_3 = 1$:

$$\mathrm{diag}(0.01, 0.01, 1).$$

By Theorem 5.3 all matrices in $\mathscr{A}$ are positively stable. $\square$

Acknowledgment

The authors wish to thank the referee for his many helpfull comments and suggestions based on which the paper has been greatly improved.

References

[1] T. Ando, Set of matrices with common Lyapunov solution, Arch. Math. 77 (2001) 76–84.
[2] J.P. Aubin, Optima and Equilibria: An Introduction to Nonlinear Analysis, Springer, Berlin, 1993.
[3] B.R. Barmish, New Tools for Robustness of Linear Systems, Macmillan, New York, 1994.
[4] A. Betser, N. Cohen, E. Zeheb, On solving the Lyapunov and Stein equations for a companion matrix, Syst. Control Lett. 25 (1995.) 211–218.
[5] S.P. Bhattacharyya, H. Chapellat, L.H. Keel, Robust Control, Prentice-Hall PTR, New Jersey, 1995.
[6] N. Cohen, I. Lewkowicz, A necessary and sufficient criterion for the stability of a convex set of matrices, IEEE Trans. Automat. Control 38 (4) (1993) 611–615.
[7] N. Cohen, I. Lewkowicz, Convex invertible cones and the Lyapunov equation, Linear Algebra Appl. 250 (1997) 105–131.
[8] N. Cohen, I. Lewkowicz, A characterization of convex cones of matrices with constant regular inertia, Linear Algebra Appl. 318 (2000) 23–33.
[9] N. Cohen, I. Lewkowicz, A pair of matrices sharing common Lyapunov solutions—a closer look, Linear Algebra Appl. 360 (2003) 83–104.
[10] L. Elsner, V. Monov, T. Szulc, On some properties of convex sets characterized by *P*-matrices and block *P*-matrices, Linear and Multilinear Algebra 50 (3) (2002) 199–218.
[11] M. Fu, B.R. Barmish, Maximal unidirectional perturbation bounds for stability of polynomials and matrices, Syst. Control Lett. 11 (1988) 173–179.
[12] D. Hershkowitz, H. Schneider, On the inertia of intervals of matrices, SIAM J. Matrix Anal. Appl. 11 (4) (1990) 565–574.
[13] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[14] C.R. Johnson, A local Lyapunov theorem and the stability of sums, Linear Algebra Appl. 13 (1976) 37–43.
[15] C.R. Johnson, L. Rodman, Convex sets of Hermitian matrices with constant inertia, SIAM J. Algebraic Discrete Methods 6 (3) (1985) 351–359.
[16] E. Kaszkurewicz, A. Bhaya, Matrix Diagonal Stability in Systems and Computation, Birkhauser Verlag AG, 1999.
[17] V.V. Monov, On the spectrum of convex set of matrices, IEEE Trans. Automat. Control 44 (5) (1999) 1009–1012.
[18] A. Nemirovskii, Several NP-Hard problems arising in robust stability analysis, Math. Control Signals Syst. 6 (1993) 99–105.
[19] S. Poljak, J. Rohn, Checking robust nonsingularity is NP-Hard, Math. Control Signals Syst. 6 (1993) 1–9.
[20] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.