On the approximation of the set of trajectories of control system described by a Volterra integral equation

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Abstract. In this paper, the set of trajectories of the control system described by a nonlinear Volterra integral equation is studied. It is assumed that the set of admissible control functions is the closed ball of the space L_p, $p > 1$, with radius μ and centered at the origin. It is shown that the sections of the set of trajectories can be approximated by the sections of trajectories, generated by the mixed constrained and Lipschitz continuous control functions, the Lipschitz constant of which is bounded.

Keywords: nonlinear Volterra integral equation, control system, integral constraint, set of trajectories, approximation.

1 Introduction

The global behaviors of many processes, which arise in various fields of science and numerous applications, are described by nonlinear integral equations (see [1–4]). Some of these processes have an exterior influences, which can be characterized as control effort or an uncertainty of the system. Therefore, mathematical models of such processes include additional functions, which are called control functions or the functions of uncertainty. In this article, such exterior influences will be accepted as control efforts. Control functions can be classified, depending on their characters, as geometric constrained control functions and integral constrained control functions. Integral constraint on controls are generally needed in modelling systems having limited energy resources, which are exhausted by consumption such as fuel or finance (see, e.g., [5–7]).

In this paper, the control systems described by a nonlinear Volterra integral equation are considered. It is assumed that the equation is nonlinear with respect to both state and control vector. The admissible control functions are chosen from the closed ball of the space L_p, $p > 1$, with radius μ and centered at the origin. The set of trajectories generated by all admissible control functions is studied. Predetermining of the properties of the set of trajectories and its numerical construction allows to predict different properties of the control system and to design required control effort. Note that the precompactness of the set of trajectories is studied in [8]. The various topological properties and numerical method for the construction of the set of trajectories are investigated in [9–11], where it is assumed that the behavior of the system is described by a nonlinear ordinary differential equation and the control functions have the same integral constraint.

The paper is organized as follows. In Section 2, the basic conditions are formulated which satisfy the system (conditions (A)–(C)). In Section 3, the set of admissible control functions is narrowed down. It is required that new control functions satisfy a geometric constraint along with an integral constraint. The evaluation of Hausdorff distance between the set of trajectories and the set of trajectories generated by mixed constrained control functions is obtained (Theorem 1). In Section 4, it is proved that the Hausdorff distance between the set of trajectories generated by the mixed constrained control functions and the set of trajectories generated by mixed constrained and Lipschitz continuous control functions is zero (Theorem 2). In Section 5, a new compact set of admissible control functions is defined. This set consists of mixed constrained and Lipschitz continuous functions, the Lipschitz constant of which is bounded. It is proved that the sections of the set of trajectories can be approximated by the sections of the set of trajectories, generated by the mixed constrained and Lipschitz continuous control functions, the Lipschitz constant of which is bounded (Theorem 4).

The Hausdorff distance between the sets $D \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is denoted by $h_n(D,E)$, where $\mathbb{R}^n$ is n-dimensional Euclidean space. The Hausdorff distance between the sets $U \subset C([a,b];\mathbb{R}^n)$ and $V \subset C([a,b];\mathbb{R}^n)$ is denoted by $h_C(U,V)$, where $C([a,b];\mathbb{R}^n)$ is the space of continuous functions $x(\cdot):[a,b] \to \mathbb{R}^n$ with norm $\|x(\cdot)\|_C = \max\{\|x(\xi)\|\colon \xi \in [a,b]\}$ and $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$.

For given $r \geqslant 0$, we set

$$
\begin{aligned}
&B_n(r) = \bigl\{x \in \mathbb{R}^n\colon \|x\| \leqslant r\bigr\}, \qquad B_n = \bigl\{x \in \mathbb{R}^n\colon \|x\| \leqslant 1\bigr\},\\
&B_C(1) = \bigl\{x(\cdot) \in C\bigl([a,b];\mathbb{R}^n\bigr)\colon \bigl\|x(\cdot)\bigr\|_C \leqslant 1\bigr\}. \qquad (1)
\end{aligned}
$$

2 Preliminaries

Consider the control system the behavior of which is described by a nonlinear Volterra-type integral equation

$$
x(\xi) = f\bigl(\xi, x(\xi)\bigr) + \lambda \int_a^\xi K\bigl(\xi, s, x(s), u(s)\bigr)\,\mathrm{d}s, \qquad (2)
$$

where $x(s) \in \mathbb{R}^n$ is the state vector of the system, $u(s) \in \mathbb{R}^m$ is the control vector, $\xi \in [a,b]$, $\lambda \geqslant 0$ is a real number.

For given $p > 1$ and $\mu > 0$, we set

$$U_p = \{u(\cdot) \in L_p([a,b];\mathbb{R}^m)\colon \|u(\cdot)\|_p \leqslant \mu\},$$

where $\|u(\cdot)\|_p = (\int_a^b \|u(s)\|^p \,\mathrm{d}s)^{1/p}$, $L_p([a,b];\mathbb{R}^m)$ is the space of Lebesgue-measurable functions $u(\cdot) : [a,b] \to \mathbb{R}^m$ such that $\|u(\cdot)\|_p < \infty$. The set $U_p \subset L_p([a,b];\mathbb{R}^m)$ is called the set of admissible control functions and every function $u(\cdot) \in U_p$ is called admissible control function.

It is assumed that the functions $f(\cdot) : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$, $K(\cdot) : [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and number $\lambda \in [0,\infty)$ given in equation (2) satisfy the following conditions:

(A) The functions $f(\cdot) : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $K(\cdot) : [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous;

(B) There exist $L_0 \in [0,1)$, $L_1 \geqslant 0$, $H_1 \geqslant 0$, $L_2 \geqslant 0$, $H_2 \geqslant 0$, $L_3 \geqslant 0$ and $H_3 \geqslant 0$ such that

$$\|f(\xi,x_1) - f(\xi,x_2)\| \leqslant L_0\|x_1 - x_2\|,$$

$$\begin{aligned}
&\|K(\xi_1,s,x_1,u_1) - K(\xi_2,s,x_2,u_2)\| \\
&\quad \leqslant [L_1 + H_1(\|u_1\| + \|u_2\|)]|\xi_1 - \xi_2| + [L_2 + H_2(\|u_1\| + \|u_2\|)]\|x_1 - x_2\| \\
&\qquad + [L_3 + H_3(\|x_1\| + \|x_2\|)]\|u_1 - u_2\|
\end{aligned}$$

for every $(\xi_1,s,x_1,u_1) \in [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m$, $(\xi_2,s,x_2,u_2) \in [a,b] \times [a,b] \times \mathbb{R}^n \times \mathbb{R}^m$;

(C) $0 \leqslant \lambda(L_2(b-a) + 2H_2(b-a)^{(p-1)/p}\mu) < 1 - L_0$.

Now let us define the trajectory of system (2) generated by an admissible control function $u(\cdot) \in U_p$. Let $u_*(\cdot) \in U_p$. A continuous function $x_*(\cdot) : [a,b] \to \mathbb{R}^n$ satisfying the equation

$$x_*(\xi) = f(\xi, x_*(\xi)) + \lambda \int_a^\xi K(\xi, s, x_*(s)u_*(s))\,\mathrm{d}s, \quad \xi \in [a,b],$$

is said to be a trajectory of system (2) generated by the admissible control function $u_*(\cdot) \in U_p$.

Conditions (A)–(C) guarantee that every admissible control function generates a unique trajectory (see [8]). We denote by $\mathbf{X}_p$ the set of all trajectories of system (2) generated by all admissible control functions $u(\cdot) \in U_p$. The set $\mathbf{X}_p$ is called the set of trajectories of system (2). According to [8], the set of trajectories $\mathbf{X}_p$ is a precompact subset of the space $C([a,b];\mathbb{R}^n)$, and hence there exists $\gamma_* > 0$ such that

$$\|x(\cdot)\|_C \leqslant \gamma_* \tag{3}$$

for every $x(\cdot) \in \mathbf{X}_p$. For each fixed $\xi \in [a,b]$, we set

$$\mathbf{X}_p(\xi) = \{x(\xi) \in \mathbb{R}^n\colon x(\cdot) \in \mathbf{X}_p\}. \tag{4}$$

It is not difficult to verify that $\mathbf{X}_p(a) = \{s_a\}$, where s_a is unique fixed point of the function $x \to f(a,x) : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $s_a \in \mathbb{R}^n$ is unique vector satisfying the equality $s_a = f(a, s_a)$.

Denote

$$R_* = \frac{\lambda(L_3 + 2\gamma_* H_3)}{1 - L_0} \exp\left[\frac{\lambda(L_2(b-a) + 2H_2\mu(b-a)^{(p-1)/p})}{1 - L_0}\right], \tag{5}$$

where γ_* is defined by (3).

The validity of the following proposition follows from conditions (A) and (B).

Proposition 1. *Let $x(\cdot) \in \mathbf{X}_p$ and $x_*(\cdot) \in \mathbf{X}_p$ be arbitrary trajectories of system* (2) *generated by the admissible control functions $u(\cdot) \in U_p$ and $u_*(\cdot) \in U_p$, respectively. Then*

$$\left\|x(\xi) - x_*(\xi)\right\| \leqslant R_* \int_a^{\xi} \left\|u(s) - u_*(s)\right\| \mathrm{d}s$$

for every $\xi \in [a,b]$.

3 The set of trajectories generated by mixed constrained control functions

For given $H \in (0, \infty)$, we set

$$U_p^H = \left\{u(\cdot) \in U_p\colon \left\|u(s)\right\| \leqslant H \text{ for every } s \in [a,b]\right\},$$

and let $\mathbf{X}_p^H$ be the set of trajectories of system (2) generated by the control functions $u(\cdot) \in U_p^H$. Let

$$\mathbf{X}_p^H(\xi) = \left\{x(\xi) \in \mathbb{R}^n\colon x(\cdot) \in \mathbf{X}_p^H\right\}, \quad \xi \in [a,b],$$

$$k_* = 2\mu^p R_*. \tag{6}$$

The following theorem characterizes the Hausdorff distance between the sets $\mathbf{X}_p$ and $\mathbf{X}_p^H$.

Theorem 1. *For each $H \in (0, \infty)$, the inequality*

$$h_C(\mathbf{X}_p, \mathbf{X}_p^H) \leqslant \frac{k_*}{H^{p-1}}$$

holds, where k_* is defined by (6).

Proof. Let us choose an arbitrary $x(\cdot) \in \mathbf{X}_p$ generated by the admissible control function $u(\cdot) \in U_p$. We define a new control function $u_0(\cdot) : [a,b] \to \mathbb{R}^m$, setting

$$u_0(s) = \begin{cases} u(s), & \|u(s)\| \leqslant H, \\ \frac{u(s)}{\|u(s)\|}H, & \|u(s)\| > H, \end{cases} \tag{7}$$

where $s \in [a,b]$. It is not difficult to verify that $u_0(\cdot) \in U_p^H$. Let $x_0(\cdot)$ be the trajectory of system (2) generated by the control function $u_0(\cdot) \in U_p^H$. Then $x_0(\cdot) \in \mathbf{X}_p^H$ and according to the Proposition 1 we have

$$\bigl\|x(\xi) - x_0(\xi)\bigr\| \leqslant R_* \int_a^{\xi} \bigl\|u(s) - u_0(s)\bigr\| \,\mathrm{d}s \tag{8}$$

for every $\xi \in [a,b]$, where R_* is defined by (5).

Now, for $\xi \in [a,b]$, we set

$$G_\xi = \bigl\{s \in [a,\xi]\colon \bigl\|u(s)\bigr\| > H\bigr\}.$$

Then we get from (7) and (8) that

$$\bigl\|x(\xi) - x_0(\xi)\bigr\| \leqslant R_* \int_{G_\xi} \bigl\|u(s) - u_0(s)\bigr\| \,\mathrm{d}s \tag{9}$$

for every $\xi \in [a,b]$.

Taking into consideration that $u(\cdot) \in U_p$, $u_0(\cdot) \in U_p^H \subset U_p$ and applying Hölder's and Minkowski's inequalities, we obtain

$$\int_{G_\xi} \bigl\|u(s) - u_0(s)\bigr\| \,\mathrm{d}s \leqslant 2\mu\bigl[\nu(G_\xi)\bigr]^{(p-1)/p}, \tag{10}$$

where $\nu(G_\xi)$ denotes the Lebesgue measure of the set G_ξ. Since $G_\xi \subset [a,b]$, $u(\cdot) \in U_p$ and $\|u(s)\| > H$ for every $s \in G_\xi$, then we have

$$H^p \cdot \nu(G_\xi) \leqslant \int_{G_\xi} \bigl\|u(s)\bigr\|^p \,\mathrm{d}s \leqslant \int_a^b \bigl\|u(s)\bigr\|^p \,\mathrm{d}s \leqslant \mu^p,$$

and hence

$$\nu(G_\xi) \leqslant \frac{\mu^p}{H^p}. \tag{11}$$

From (6), (9), (10) and (11) it follows that

$$\bigl\|x(\xi) - x_0(\xi)\bigr\| \leqslant \frac{2\mu^p}{H^{p-1}} R_* = \frac{k_*}{H^{p-1}}$$

for every $\xi \in [a,b]$ and consequently,

$$\bigl\|x(\cdot) - x_0(\cdot)\bigr\|_C \leqslant \frac{k_*}{H^{p-1}}.$$

Since $x(\cdot) \in \mathbf{X}_p$ is arbitrarily chosen, we conclude that

$$\mathbf{X}_p \subset \mathbf{X}_p^H + \frac{k_*}{H^{p-1}} B_C(1), \tag{12}$$

where $B_C(1)$ is defined by (1). From inclusion $\mathbf{X}_p^H \subset \mathbf{X}_p$ and (12) we obtain the proof. $\square$

4 The set of trajectories generated by the mixed constrained and Lipschitz continuous control functions

Let us define new set of control functions, setting

$$U_p^{H,lip} = \{u(\cdot) \in U_p^H\colon\ u(\cdot) : [a,b] \to \mathbb{R}^m \text{ is Lipschitz continuous}\},$$

and let $\mathbf{X}_p^{H,lip}$ be the set of trajectories of system (2) generated by the control functions $u(\cdot) \in U_p^{H,lip}$. Denote

$$\mathbf{X}_p^{H,lip}(\xi) = \{x(\xi) \in \mathbb{R}^n\colon\ x(\cdot) \in \mathbf{X}_p^{H,lip}\}, \quad \xi \in [a,b], \tag{13}$$

$$g_0 = R_*\big(b-a\big)^{(p-1)/p}, \tag{14}$$

where R_* is defined by (5).

The following theorem characterizes the Hausdorff distance between the sets $\mathbf{X}_p^H$ and $\mathbf{X}_p^{H,lip}$.

Theorem 2. *Let $H > 0$ be fixed. Then*

$$h_C\big(\mathbf{X}_p^H, \mathbf{X}_p^{H,lip}\big) = 0.$$

Proof. Let us choose an arbitrary $\varepsilon > 0$. Now let us choose an arbitrary $x(\cdot) \in \mathbf{X}_p^H$ and let $u(\cdot) \in U_p^H$ be the control function, generating the trajectory $x(\cdot)$. For given $h \in (0,1)$, let $u_h(\cdot)$ be the Steklov function of $u(\cdot) \in U_p^H$, i.e.,

$$u_h(s) = \frac{1}{2h} \int\limits_{s-h}^{s+h} \widetilde{u}(\tau)\, \mathrm{d}\tau, \quad s \in [a,b],$$

where

$$\widetilde{u}(\tau) = \begin{cases} u(\tau), & \tau \in [a,b], \\ 0, & \tau \in [a-1,a) \cup (b,b+1]. \end{cases}$$

$u(\cdot) \in U_p^H$ yields that $\|u(\cdot)\|_p \leqslant \mu$ and $\|u(s)\| \leqslant H$ for every $s \in [a,b]$. According to [12], we have that, for each fixed $h \in (0,1)$, the inequalities $\|u_h(\cdot)\|_p \leqslant \mu$ and $\|u_h(s)\| \leqslant H$ are satisfied for every $s \in [a,b]$, the function $u_h(\cdot) : [a,b] \to \mathbb{R}^m$ is Lipschitz continuous with Lipschitz constant H/h. Then we obtain that $u_h(\cdot) \in U_p^{H,lip}$ for each fixed $h \in (0,1)$. Moreover, it is known (see, e.g., [12]) that $\lim_{h\to 0^+} \|u_h(\cdot) - u(\cdot)\|_p = 0$. Then, for given $\varepsilon/g_0 > 0$, there exists $h_0 \in (0,1)$ such that

$$\big\|u_{h_0}(\cdot) - u(\cdot)\big\|_p \leqslant \frac{\varepsilon}{g_0}, \tag{15}$$

where $g_0 > 0$ is defined by (14). Let $u_0(\cdot) = u_{h_0}(\cdot)$ and let $x_0(\cdot)$ be the trajectory of system (2) generated by the control function $u_0(\cdot)$. Thus, $x_0(\cdot) \in \mathbf{X}_p^{H,lip}$. Proposition 1,

(14) and (15) imply that

$$\begin{aligned}\|x(\xi)-x_0(\xi)\| &\leqslant R_* \int_a^{\xi} \|u(s)-u_0(s)\| \, \mathrm{d}s \\ &\leqslant R_*(b-a)^{(p-1)/p} \|u(\cdot)-u_0(\cdot)\|_p \leqslant \varepsilon\end{aligned}$$

for every $\xi \in [a,b]$ and hence

$$\|x(\cdot)-x_0(\cdot)\|_C \leqslant \varepsilon. \tag{16}$$

Thus, for each $x(\cdot) \in \mathbf{X}_p^H$, there exists $x_0(\cdot) \in \mathbf{X}_p^{H,lip}$ such that inequality (16) is verified. This means that

$$\mathbf{X}_p^H \subset \mathbf{X}_p^{H,lip} + \varepsilon B_C(1). \tag{17}$$

Taking into consideration that $\mathbf{X}_p^{H,lip} \subset \mathbf{X}_p^H$, we obtain from (17) that

$$h_C(\mathbf{X}_p^H, \mathbf{X}_p^{H,lip}) \leqslant \varepsilon. \tag{18}$$

Since $\varepsilon > 0$ is an arbitrarily chosen, (18) completes the proof. □

In [8], it is shown that the set of trajectories $\mathbf{X}_p$ in general is not closed. It is also possible to show that the set $\mathbf{X}_p^H$ is not closed one in general. Therefore, from Theorem 2 we get the validity of the equality

$$\mathrm{cl}(\mathbf{X}_p^H) = \mathrm{cl}(\mathbf{X}_p^{H,lip}),$$

where cl denotes the closure of a set.

5 Approximation of the sections of the set of trajectories by the compact sets

For given $\mathcal{L} > 0$, we denote

$$U_p^{H,lip,\mathcal{L}} = \{u(\cdot) \in U_p^{H,lip} \colon u(\cdot) : [a,b] \to \mathbb{R}^m \text{ is Lipschitz continuous and } \\ \text{Lipschitz constant is not greater than } \mathcal{L}\},$$

and let $\mathbf{X}_p^{H,lip,\mathcal{L}}$ be the set of trajectories of system (2) generated by the control functions $u(\cdot) \in U_p^{H,lip,\mathcal{L}}$. For given $\xi \in [a,b]$, we set

$$\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi) = \{x(\xi) \in \mathbb{R}^n \colon x(\cdot) \in \mathbf{X}_p^{H,lip,\mathcal{L}}\}. \tag{19}$$

Proposition 2. *For each fixed $H > 0$ and $\mathcal{L} > 0$, the set of trajectories $\mathbf{X}_p^{H,lip,\mathcal{L}}$ is a compact subset of the space $C([a,b];\mathbb{R}^n)$ and, for each $\xi \in [a,b]$, the set $\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi)$ is a compact subset of the space $\mathbb{R}^n$.*

Proof. According to [8], the set of trajectories $\mathbf{X}_p$ is a precompact subset of the space $C([a,b];\mathbb{R}^n)$. Since $\mathbf{X}_p^{H,lip,\mathcal{L}} \subset \mathbf{X}_p$, then we have that the set $\mathbf{X}_p^{H,lip,\mathcal{L}}$ is a precompact subset of the space $C([a,b];\mathbb{R}^n)$. From compactness of the set of control functions $U_p^{H,lip,\mathcal{L}}$ in the space $C([a,b];\mathbb{R}^n)$ it follows that the set of trajectories $\mathbf{X}_p^{H,lip,\mathcal{L}}$ is a closed subset of the space $C([a,b];\mathbb{R}^n)$, and hence it is a compact set. $\square$

Proposition 3. *Let $H > 0$ be fixed. Then, for each $\xi \in [a,b]$, the equality*

$$\lim_{\mathcal{L}\to\infty} \mathbf{X}_p^{H,lip,\mathcal{L}}(\xi) = \mathrm{cl}\left(\mathbf{X}_p^{H,lip}(\xi)\right) \tag{20}$$

holds, and hence, for given $\varepsilon > 0$, there exists $\mathcal{L}_*(\xi,\varepsilon,H) > 0$ such that the inequality

$$h_n\left(\mathbf{X}_p^{H,lip}(\xi), \mathbf{X}_p^{H,lip,\mathcal{L}}(\xi)\right) < \varepsilon \tag{21}$$

is verified for every $\mathcal{L} \geqslant \mathcal{L}_*(\xi,\varepsilon,H)$.

Here the sets $\mathbf{X}_p^{H,lip}(\xi)$ and $\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi)$ are defined by (13) *and* (19)*, respectively, $\lim_{\mathcal{L}\to\infty} \mathbf{X}_p^{H,lip,\mathcal{L}}(\xi)$ is Kuratowski limit of the set sequence $\{\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi)\}_{\mathcal{L}=1}^{\infty}$ (see* [13]*).*

Proof. From equality $U_p^{H,lip} = \bigcup_{\mathcal{L}=1}^{\infty} U_p^{H,lip,\mathcal{L}}$ we obtain that $\mathbf{X}_p^{H,lip} = \bigcup_{\mathcal{L}=1}^{\infty} \mathbf{X}_p^{H,lip,\mathcal{L}}$, and hence

$$\mathbf{X}_p^{H,lip}(\xi) = \bigcup_{\mathcal{L}=1}^{\infty} \mathbf{X}_p^{H,lip,\mathcal{L}}(\xi) \tag{22}$$

for each $\xi \in [a,b]$. Since $\mathbf{X}_p^{H,lip,\mathcal{L}} \subset \mathbf{X}_p$ for each $H > 0$ and $\mathcal{L} = 1,2,\dots$, then from (3) we have that $\|x(\cdot)\|_C \leqslant \gamma_*$ for every $x(\cdot) \in \mathbf{X}_p^{H,lip,\mathcal{L}}$. Thus, for each fixed $\xi \in [a,b]$, the inclusion

$$\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi) \subset B_n(\gamma_*)$$

is satisfied for every $H > 0$ and $\mathcal{L} = 1,2,\dots$. Since

$$\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi) \subset \mathbf{X}_p^{H,lip,\mathcal{L}+1}(\xi) \subset B_n(\gamma_*) \tag{23}$$

for every $\mathcal{L} = 1,2,\dots$, then it is not difficult to verify that (22) and (23) imply the validity of equality (20). $\square$

Thus, according to Proposition 3 for each fixed $\varepsilon > 0$, $H \in (0,\infty)$ and $\xi \in [a,b]$, there exists $\mathcal{L}_*(\xi,\varepsilon,H) > 0$ such that inequality (21) is satisfied for every $\mathcal{L} > \mathcal{L}_*(\xi,\varepsilon,H)$. Is it possible to choose the number $\mathcal{L}_*(\xi,\varepsilon,H) > 0$ not depending on ξ? The answer of this question is positive.

Theorem 3. *Let $H > 0$ be fixed. Then, for each $\varepsilon > 0$, there exists $\mathcal{L}(\varepsilon,H) > 0$ such that, for every $\mathcal{L} \geqslant \mathcal{L}(\varepsilon,H)$ and $\xi \in [a,b]$, the inequality*

$$h_n\left(\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi), \mathbf{X}_p^{H,lip}(\xi)\right) < \varepsilon$$

is satisfied.

The proof of the theorem follows from Proposition 3 and Proposition 3 of [8], where it is proved that there exists a function $\varphi(\cdot) : (0,+\infty) \to (0,+\infty)$ such that $\varphi(\delta) \to 0$ as $\delta \to 0^+$ and

$$\big\|x(\xi^*) - x(\xi_*)\big\| \leqslant \varphi\big(|\xi^* - \xi_*|\big)$$

for every $\xi^* \in [a,b]$, $\xi_* \in [a,b]$ and $x(\cdot) \in \mathbf{X}_p$.

The following theorem gives us an evaluation between the sets $\mathbf{X}_p(\xi)$ and $\mathbf{X}_p^{H,lip,\mathcal{L}}(\xi)$, where the sets $\mathbf{X}_p(\xi)$ and $\mathbf{X}_p^{H(\varepsilon),lip,\mathcal{L}}(\xi)$ are defined by (4) and (19), respectively.

Theorem 4. *For each $\varepsilon > 0$, there exist $H(\varepsilon) > 0$ and $\mathcal{L}(\varepsilon) = \mathcal{L}(\varepsilon, H(\varepsilon)) > 0$ such that, for every $\mathcal{L} > \mathcal{L}(\varepsilon)$, the inequality*

$$h_n\big(\mathbf{X}_p(\xi), \mathbf{X}_p^{H(\varepsilon),lip,\mathcal{L}}(\xi)\big) < \varepsilon$$

is verified for every $\xi \in [a,b]$.

Proof. For given $\varepsilon > 0$, we denote

$$H(\varepsilon) = \left(\frac{2k_*}{\varepsilon}\right)^{1/(p-1)}, \tag{24}$$

where k_* is defined by (6). From Theorem 1 and (24) it follows that

$$h_n\big(\mathbf{X}_p(\xi), \mathbf{X}_p^{H(\varepsilon)}(\xi)\big) \leqslant \frac{k_*}{H(\varepsilon)^{p-1}} = \frac{\varepsilon}{2} \tag{25}$$

for every $\xi \in [a,b]$. Theorem 2 implies that, for fixed $H(\varepsilon) > 0$, the equality

$$h_n\big(\mathbf{X}_p^{H(\varepsilon)}(\xi), \mathbf{X}_p^{H(\varepsilon),lip}(\xi)\big) = 0 \tag{26}$$

holds for every $\xi \in [a,b]$. By virtue of Theorem 3 for $H = H(\varepsilon)$ there exists $\mathcal{L}(\varepsilon) = \mathcal{L}(\varepsilon, H(\varepsilon))$ such that, for every $\mathcal{L} > \mathcal{L}(\varepsilon)$, the inequality

$$h_n\big(\mathbf{X}_p^{H(\varepsilon),lip}(\xi), \mathbf{X}_p^{H(\varepsilon),lip,\mathcal{L}}(\xi)\big) < \frac{\varepsilon}{2} \tag{27}$$

is satisfied for every $\xi \in [a,b]$. (25), (26) and (27) give us the proof of the theorem. $\square$

Theorem 4 enables one to give an approximation of graph of the set valued map $\xi \to \mathbf{X}_p(\xi)$, $\xi \in [a,b]$, where the graph of the set valued map $\xi \to \mathbf{X}_p(\xi)$, $\xi \in [a,b]$, is denoted by $\operatorname{gr}\mathbf{X}_p(\cdot)$ and is defined as

$$\operatorname{gr}\mathbf{X}_p(\cdot) = \big\{(\xi, x) \in [a,b] \times \mathbb{R}^n\colon\ x \in \mathbf{X}_p(\xi)\big\}.$$

Let $\varGamma = \{a = \xi_0 < \xi_1 < \cdots < \xi_N = b\}$ be a partition of the closed interval $[a,b]$. The diameter of $\varGamma$ is denoted by $\operatorname{diam}(\varGamma)$ and is defined as

$$\operatorname{diam}(\varGamma) = \max\{\xi_{i+1} - \xi_i\colon\ i = 0, 1, \ldots, N-1\}.$$

For given partition $\Gamma = \{a = \xi_0 < \xi_1 < \cdots < \xi_N = b\}$ of the closed interval $[a, b]$, we set

$$\mathbf{Z}_p^{H,lip,\mathcal{L},\Gamma} = \bigcup_{i=0}^{N} \big(\xi_i, \mathbf{X}_p^{H,lip,\mathcal{L}}(\xi_i)\big).$$

The following corollary characterizes the Hausdorff distance between the sets $\mathrm{gr}\,\mathbf{X}_p(\cdot)$ and $\mathbf{Z}_p^{H,lip,\mathcal{L},\Gamma}$.

Corollary 1. *For each $\varepsilon > 0$, there exists $H(\varepsilon) > 0$, $\mathcal{L}(\varepsilon) = \mathcal{L}(\varepsilon, H(\varepsilon)) > 0$, $\delta(\varepsilon) > 0$ such that, for every $\mathcal{L} > \mathcal{L}(\varepsilon)$ and partition Γ of the closed interval $[a, b]$ with $\mathrm{diam}(\Gamma) < \delta(\varepsilon)$, the inequality*

$$h_{n+1}\big(\mathrm{gr}\,\mathbf{X}_p(\cdot), \mathbf{Z}_p^{H(\varepsilon),lip,\mathcal{L},\Gamma}\big) < \varepsilon$$

is satisfied. Here $h_{n+1}(\cdot,\cdot)$ denotes Hausdorff distance between the sets of the space $\mathbb{R}^{n+1}$.

The proof of the corollary follows from Theorem 4 and Proposition 3 of [8].

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