Croatian Journal of Education
Vol.21; No.4/2019, pages: 1133-1179
Professional paper
Paper submitted: 02nd May 2018
Paper accepted: 08th February 2019
https://doi.org/10.15516/cje.v21i4.3205

A Lesson Study to Develop Teachers' Geometric Habits of Mind

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Abstract

The general purpose of this study is to investigate the development of geometric thinking of elementary school mathematics teachers. To achieve this, the model of research is defined as a lesson study. The implementation process of the study was carried out with five mathematics teachers from different schools. Firstly, a five-week seminar process was carried out with teachers. As part of this process, the lesson study was explained with the theoretical framework of Geometric Habits of Mind and its practices were implemented. Then a lesson study was performed which took about three months. Data was collected by using multiple data collection tools and data analysis was carried out by using a video analysis model developed by Powell, Francisco, and Maher (2003). As a result of this study, geometric thinking of teachers has improved through the lesson study. During the process from the first lesson study to the seventh lesson study, we observed the improvement in the mathematical language used by teachers, their explanations, student questions in class, together with activities and problems produced by teachers based on the geometric habits of mind for the relevant concepts. In the instructional processes, we also observed the teachers considering and assessing these components and planning for and implementing them in the geometry lessons. After the lesson studies, teachers also considered the geometric habits in their own school settings and reflected on the instructional process through prepared activities and problems which were then identified.

Keywords: *geometric habits of mind; geometric thinking; lesson study; teacher professional development*

Introduction

Geometry can enable individuals to analyse and solve problems, and also allow them to make connections between mathematics and life since it provides individuals with an opportunity to interpret the physical environment. Geometry is also used as a tool for studying science and other science-related subjects (Clements & Battista, 1992) and gives individuals a perspective. It is important for individuals to possess knowledge of geometry at a specific level, for example, to be able to easily make geometric measurements or to have geometric thinking skills, particularly when encountering real-life problems. These skills will make their daily lives easier. Geometric thinking has been defined as an "ability to think about geometric situations and to make inferences" (Van de Walle, 2004). On the basis of this definition, it is possible to say that geometric thinking is developed through geometric reasoning, geometric relationships, and is also an ability to make inferences based on these generalizations.

Geometry instruction is a process that starts during pre-school education and continues throughout secondary education. In elementary schools, students' knowledge of geometry, which is already intuitive, must be conceptualised and developed through tangible models (Ersoy, 2006). In this way, children can apply their knowledge of geometry to the problems they encounter in real life. Although the main purpose of geometry instruction is identified as "a student's ability to use geometry during the problem-solving process and when explaining his/her own physical world, environment and the universe," the Euclidean Geometry used in existing mathematical programs is not sufficient to help students understand their environment and to establish relationships with the environment in which they live (Baki, 2001). Therefore, research on the teaching process is necessary in order to make students aware of geometric concepts and to allow them to find meaning in these concepts (Clements & Battista, 1992). The education the teachers provide aims to equip the students with geometric knowledge and skills. This must not be limited to simply providing them with geometric knowledge, but it also must ensure that they are equipped with geometric reasoning skills and the ability to think geometrically. Therefore, the teachers' aim must be to develop geometric thinking in addition to equipping them with the knowledge and skills included in the program.

In order for Mathematics teachers to provide good geometry instruction, it is necessary for them to have a deep understanding of geometry (Jones, 2000). Research shows that in-service and pre-service Mathematics teachers have insufficient knowledge of geometry (Fuys, Geddes & Tischler, 1988; Swafford, Jones & Thornton, 1997; Mayberry, 1983); the mistakes made by Mathematics teachers show parallels with the mistakes made by students (Herskowitz & Vinner, 1984; Swafford, Jones & Thornton, 1997), and teachers' knowledge of geometry has a significant effect upon their students' learning in the subject (Lenhart, 2010; Clements, 2003). If teachers have a good understanding of geometry, progress can be made in understanding the difficulties which the students encounter. Consequently, solutions to the difficulties encountered in the teaching process can be sought (Durmuş, Toluk & Olkun, 2002). In order to develop teachers' knowledge of geometry and their geometry instruction, they must have sufficient knowledge and experience (Toluk, Olkun & Durmuş, 2002). Although it has been emphasized that effective learning environments can only be provided by experienced teachers (Putnam, Heaton, Prawat & Remillard, 1992), it is also the case that teachers' knowledge should be addressed during in-service training (Aslan-Tutak, 2011), since there is evidence that they do not have any background in the field of geometry and their pedagogical knowledge in geometry is insufficient to teach the subject (Jones, 2000).

As the teachers' knowledge of geometry and the students' awareness of their own cognitive processes improve, changes in what and how they teach can be observed (Swafford, Jones & Thornton, 1997; Mistretta, 2000). When one considers the results produced by a teacher's knowledge level, it is necessary for Mathematics teachers to have both a good knowledge of geometry and to be able to teach it, taking into consideration the students' cognitive processes regarding geometry (Toluk, Olkun & Durmuş, 2002). Although teachers must possess a working knowledge (Tanışlı, 2013), researchers have emphasized that having sufficient knowledge does not mean that Mathematics teachers will be successful in instructing, and that individuals should synthesize their mathematical knowledge and knowledge about instruction (Kahan, Cooper & Bethea, 2003; Türnüklü, 2005). The experiences gained by the recent research done and the instructions carried out show that when students have an in-depth knowledge of geometry and geometric reasoning, they are able to overcome the difficulties they encounter while learning other fields of mathematics. Therefore, teachers must be conscious of the importance of geometry within the mathematical program, and researchers must provide those teachers with the knowledge and tools which will help them to develop their own activities (Gutiérrez, 2014). The American National Council of Teachers of Mathematics– NCTM also emphasizes that teachers are required to help their students in promoting geometric reasoning, and to increase their knowledge regarding concepts, in addition to their procedural knowledge (NCTM, 2000). The objective of this research is to apply a lesson study model in the context of developing geometric thinking among teachers. This is based on the idea that teachers play a key role in developing geometry instruction and geometric thinking.

Theoretical Framework

When the research on individuals' geometric thinking development is reviewed, it can be seen that there are two important theories which have been suggested, mainly in the research of Piaget et al. (Piaget & Inhelder, 1956; Piaget, Inhelder & Szeminska, 1964) and van Hiele (1986). Although research on geometry dates back a long way, the first known research regarding the development of geometric thinking was published in a book called "The Child's Conception of Space" written by Piaget and Inhelder (1956) in 1948. Although this research has attracted considerable attention over the years, it has faced heavy criticism in recent years in terms of the way it addresses assumptions of mathematical definitions and the way it views children's improvements in geometric thinking (Bjorklund, 1997; Clements & Battista, 1992; Darke, 1982; Geeslin & Shar, 1979; Kapadia, 1974; Laurendau & Pinard, 1970; Martin, 1976; Peel, 1959; Somerville & Bryant, 1985; van Der Sandt, 2000).

Following the research of Piaget and Inhelder (1956), one piece of research which attracted considerable attention regarding geometric thinking was carried out by the Dutch educationalists, Pierre Marie van Hiele and Dina van Hiele-Geldof. In this research, the formation of geometric concepts and the development of geometric thinking in children have been examined. Through classifying by levels (Visualization/Recognition, Analysis, Informal Deduction/Order, Formal Deduction, and Rigor), the developmental stages of geometric thinking were systemised, based on the abilities of individuals starting from the level of sorting figures by their similarity, differences and appearances, up to the level of relating various systems to spaces (Van Hiele, 1986). According to this model, the van Hiele test is used to identify geometric thinking in individuals. Some researchers (Manizade, 2006; Lenhart, 2010) have indicated that although different versions of the van Hiele test are used for reliable evaluation of teachers' knowledge of geometry (Usiskin & Senk, 1990; Wilson, 1990), this test does not evaluate their pedagogical knowledge. Therefore, in the processes of developing the geometric thinking of teachers and analysing this development, the "Geometric Habits of Mind – GHoM" was used in this research, which is quite a new theoretical framework when compared to the approaches mentioned above.

The framework of the *Geometric Habits of Mind* which has been suggested by Driscoll, Wing DiMatteo, Nikula, and Egan (2007), and which plays a significant role in the development of geometric thinking, is a model developed from the framework of the *Mathematical Habits of Mind-MHoM* (Cuoco, Goldenberg & Mark, 1996). The *Mathematical Habits of Mind* have been defined as special ways of thinking, which are similar to the methods mathematicians use in their problem-solving approaches and for thinking about mathematical concepts (Cuoco, Goldenberg & Mark, 1996; 2010). In addition, the *Mathematical Habits of Mind* have also been defined as efficient ways of thinking which promote learning, and the application and understanding of formal mathematics (Driscoll et al., 2007). Goldenberg, Cuoco, and Mark (1998) indicated that mathematical power can be ideally defined using the set of Mathematical Habits. This concept is a principle of organization that has been suggested by Cuoco, Goldenberg, and Mark (1996) for a mathematical program in which high school students and university students think about the methods which mathematicians use. The *Mathematical Habits of Mind* have been suggested in order to enable students to understand mathematicians' ways of thinking (Lim & Selden, 2009). The aim of these habits has been specified as "helping students learn and adopt mathematicians' ways of thinking about problems" (Cuoco, Goldenberg & Mark, 1996; Lim & Selden, 2009).

The *Geometric Habits of Mind (GHoM)* have been developed within the context of the development of the geometric thinking project on the grounds of the previous research which has been done by Driscoll et al. regarding the *Mathematical Habits of*

Mind, and it is a framework that emphasizes efficient ways of thinking, which have been specifically grounded in geometric thinking. When compared to the framework of the *Mathematical Habits of Mind*, this framework includes components unique to geometric thinking. The researchers who worked on this framework of four interrelated *Geometric* Habits (Driscoll, Wing DiMatteo, Nikula, Egan, Mark & Kelemanik, 2008) indicated that it is based on the perspective of geometric thinking. The theoretical framework of *GHoM* helps teachers see whether or not students are thinking during the process of solving geometric problems, thanks to the means by which it includes efficient ways of geometric problem-solving. This framework focuses on determining the evidence for geometric thinking. Within the theoretical framework, ways for students and adults to become successful geometric problem-solvers have been identified, and analyses of geometric thinking evidence have been included (Köse & Tanışlı, 2014).

The four main components of the *Geometric Habits of Mind* have been defined as *Reasoning with Relationships, the Generalizing Geometric Ideas habit of mind, Investigating Invariants* and *Balancing Exploration and Reflection* (Driscoll et al., 2008).

The *Reasoning with Relationships habit of mind* can be defined as "actively looking for relationships within geometric figures and among geometric figures (for instance, accompaniment and similarity)" (Driscoll et al., 2008). These relationships may be among separate figures, among a figure's own pieces, in a figure as a whole, or among concepts (such as area and circumference). The questions revealing this habit have been defined as *Are these figures alike? How? In how many ways will they be alike? Are these figures different from each other? How? What should I have done to make this object appear similar to the other one?* (Driscoll et al., 2008). This habit focuses on relationships among different figures, a figure or sub-figures within a figure, and also includes some reasoning skills in which proportion and symmetry are used.

The *Generalizing Geometric Ideas habit of mind* can be defined as "wanting to understand and to describe the '*frequently*' and the '*always*' expressions which are related to the geometric concepts and operations" (Driscoll et al., 2008). *Generalizing Geometric Ideas habit of mind* is achieved through stages of making conjectures regarding *usually, each* and *how many cases* expressions; testing the conjectures; making inferences about the conjectures; providing a convincing defence which will reinforce this result. The questions revealing this habit may be, for example: *Does it happen in every case? Why would this happen in every case? Have I found all the ones that fit this description? Can I think of examples when this is not true, and, if so, should I then revise my generalisation? Why would this apply in other dimensions?* (Driscoll et al., 2008).

Investigating Invariants habit of mind can be defined as "analysing which properties of a geometric figure are affected as a result of a transformation (such as reflection, parallel displacement, rotation)" (Driscoll et al., 2008). Here, the invariants have been defined as "the properties which always stay the same even if the other states change in a figure." Unchanging properties of a figure during a transformation include its area, circumference, volume, side length, and the ratio of side lengths and angles. The questions revealing this habit have been defined as *How can we move it from there to here? What has changed? Why? What things have not changed? Why?* (Driscoll et al., 2008). As the researchers have conducted studies on this habit, they have shown that as the students think about the *Investigating Invariants* component, they understand the situations when they pay attention to the effect of the transformation made in the figures.

Table 1

Conceptual Framework of the Geometric Habits of Mind

Reasoning with Relationships	*Focus on relationships among separate figures, by…* • comparing two or more geometric figures by enumerating some properties they have in common • contrasting two or more geometric figures by noting properties they do not have in common • comparing two or more geometric figures by considering relationships for their one-dimensional, two-dimensional, or three-dimensional components	*Focus on relationships among the pieces in a single figure, by…* • noticing and relating subfigures within a geometric figure • constructing subfigures within a geometric figure • relating two geometric figures by noticing they can be seen as parts of a single geometric figure	*Use spatial reasoning skills to focus on relationships, by…* • reasoning proportionally about two or more geometric figures • using symmetry to relate geometric figures
Generalizing Geometric Ideas	*Seek solutions from familiar cases or known solutions, by…* • considering relevant special cases (such as the side lengths which are a whole number) • looking beyond special cases to some other examples that fit (*Trying a side length which is not a whole number*) • generating new cases by changing features in cases already identified *(reflection, parallel displacement)* • intuiting that there are other solutions, without knowing how to generate them	*Seek a range of solutions using assumed simplifying conditions, by…* • recognizing that the given conditions work for an infinite set, but considering only a discrete set • seeing an infinite, continuously varying set of cases that work, but limiting the set or jumping to the wrong conclusion about the set, or • reaching an incorrect conclusion about the set	*Seek complete solution sets or general rules, by…* • seeing the entire set of solutions and explaining why there are no more • noticing a rule that is universally true for a class of geometric figures • situating problems or rules in broader contexts
Investigating Invariants	*Use dynamic thinking and searching, by…* • thinking dynamically about a static case • wondering about what changes and what stays the same when transformation is applied • generating a number of cases of transformation effects and looking for commonalities • thinking about the effects of moving a point or figure continuously and predicting occurrences in between one point and another • considering limit cases and extreme cases under transformations	*Check evidence of effects, by…* • intuiting that not everything is changing as a transformation is applied • noticing that the same effect appears to happen each time a particular type of transformation is applied • noticing invariants when a transformation is applied and explaining why there are invariants	
Balancing Exploration and Reflection	*Put exploration in the foreground by…* • drawing, playing, and/or exploring through intuition or guessing • drawing, playing, and/or exploring with regular stocktaking • considering previous similar situations • changing or considering changes to some feature of situation, condition, or geometric figure	*Put end goals in the foreground by…* • periodically returning to the big picture as a touchstone of progress • identifying intermediate steps that can help get to the goal • describing what the final state would look like • making reasoned conjectures about solutions, creating ways to test conjectures	

Balancing Exploration and Reflection habit of mind can be defined as "thinking of trying to adopt different approaches which were usually selected as the result of the offered hypotheses, and on a regular basis taking into consideration what has been learned" (Driscoll et al., 2008). It is important, as far as possible; to keep a balance between the explorations supported by the hypotheses and the reflection of what has been learned as a result of these explorations. The questions revealing this habit may be such as: *What will happen if I draw a picture, and/or if I add or remove this picture, and/or if I begin to think from the point of view of the consequences and in a reverse way, etc.? What does this action mean to me? How can my first attempts to solve this problem explain the current approach I adopt?* (Driscoll et al., 2008).

In brief, *Reasoning with Relationships* is thinking about the geometric figures through researching geometric relationships and using spatial reasoning skills; *Generalizing Geometric Ideas* is understanding and identifying geometric facts; *Investigating Invariants* is researching changing and unchanging states and properties in a geometric framework, and *Balancing Exploration and Reflection* is trying to use different ways of solving a problem and going back to the first steps to review the process on a regular basis. These habits and their components are defined in Table 1 in the way that was used in the research of Driscoll et al. (2008). In addition to the research of Driscoll et al. (2007), there are two more pieces of research in the national literature related to this subject. These are the research of Bülbül (2016) in which pre-service mathematics teachers' knowledge has been evaluated, the lesson study of Özen and Köse (2013) about geometric objects which has been applied to the mathematics teachers, and the research of Köse and Tanışlı (2014) which has set forth the pre-service primary teachers' geometric habits.

The objective of this research was to examine the development of geometric thinking in secondary school mathematics teachers by applying a lesson study. In accordance with this purpose, this study was intended to provide an answer to the question of *How does secondary school mathematics teachers' geometric thinking develop within the scope of GHoM during the implementation process of the lesson study model?*

Methods

When research studies on the teachers' professional development are examined, it can be observed that the "lesson study" takes its place among popular approaches which are adopted, particularly for mathematics teachers' professional development (Stigler & Hiebert, 1999; Lewis, Perry, Hurd & O'Connell, 2006; Watanebe, 2002; Fernandez, 2002; Saito, 2012; Lewis, Perry, Friedkin & Roth, 2012). This is a professional development model which enables participants to see the in-class implementations from a different perspective (Stigler & Hiebert, 1999), and it is also identified as "a collaborative professional development approach for teachers" (Murata, 2011; Lewis & Tsuchida, 1998; Stigler & Hiebert, 1999). This approach includes the stages in which the teachers come together and then plan, apply, and evaluate an efficient lesson collaboratively, which will enable the students to learn (Murata, 2011; Baki, Baki & Arslan, 2011). Among the educational survey models, the "lesson study" model, which is a qualitative research method, was used in this research with the aim of developing the geometric thinking levels of secondary school teachers in their own school environments.

The lesson study model begins with a stage in which teachers come together and designate the end goals that will ensure the learning and the development of the students (Lewis, Perry & Murata, 2006). Stigler and Hiebert (1999) have identified the stages of a lesson study as defining the problem, planning a lesson, conducting a research lesson, reflecting and evaluating the lesson, revising the lesson, re-teaching the revised lesson, and sharing the re-taught lesson's reflections, evaluation and results. The most outstanding part of these stages is the research lesson in which the planned lesson is taught and monitored in a real class environment. These are lessons which are conducted with the participating teacher's students, and which are different from their daily class environments, have unique properties, and are conducted in a real class atmosphere (Lewis & Tsuchida, 1998). After the goals of the lesson are defined during the preparation process, the teachers prepare a lesson plan. Rather than trying to achieve a good plan, the aim of this stage is to evaluate the way in which the students learn and how the teaching approach is adopted, or to explore a question about the learning (Murata, 2011).

The lesson study model puts the research lesson in the centre of the process (Murata, 2011). The research studies show that Japanese teachers state that applied courses are the most important contributor to their professional development (Murata, 2011; Murata & Takahashi, 2002). Through the research lessons, the teachers see the education models and how they affect the learning of a student (Murata, 2011). The research lessons serve the purpose of developing classroom practice, disseminating new content and its approaches, connecting classroom practices with more comprehensive gains, exploring contradictory ideas and providing a different approach, shaping national policies, and appreciating the role of education provided in the classroom (Lewis & Tsuchida, 1998). The research lessons are conducted in real classroom environments, and they give teachers a special learning opportunity, which is otherwise impossible for them to have within the developed professional development community (Murata, 2011). The in-class teaching implementations are examined as a whole, rather than watching some parts of videotape recordings or reading some parts of books about the education process (Murata, 2011).

In the lesson study model, which has a circular approach as the cycle repeats itself a few times, the teachers have an opportunity to discuss the learning and how the teaching provided affects the learning (Murata, 2011). This circular structure is given below (Figure 1). The fact that the nature of the model shows the teachers' geometric thinking and provides them with an opportunity to create an environment in which they can utilize each other's geometric thinking, was the reason why the lesson study model was preferred in this research.

Figure 1. The Lesson Study Cycle

Study Group

The study group of this research consisted of volunteer Mathematics teachers from different secondary schools in Aydın, Turkey. This research was conducted by working with a total of eight teachers; three of them were included during the pilot study process, and five of them were included during the implementation process. In both processes, the research based on voluntary participation was conducted by working with secondary school Mathematics teachers selected from the teachers whose weekly course schedules were convenient for the timescale of the implementation of the study. The teachers who participated in the pilot study were excluded from the main implementation.

With the aim of selecting the study group of the main research, the researcher held meetings with teachers from all secondary schools in the central district, and at this point, they were informed about the research process. Meanwhile, teachers who wanted to participate voluntarily in the research were identified and a contact meeting was conducted with them. The lesson study model which would be used in the research for the teachers' professional development and the theoretical framework of *Geometric Habits of Mind* which would be used as a base for the development of geometric thinking was explained to those 14 teachers who participated in this meeting. In addition, some examples regarding the pilot study were given and the draft work schedule of the research was presented. Then, out of those 14 teachers, five teachers were identified as the participants of the research and were given further explanations of how they were required to participate in the teacher seminars, which would be organized in line with the end goals of the research, to participate in the lesson study activities to be conducted, and also to perform their own tasks in the group within the prescribed time. The study group was informed about the implementation and notified in writing of the tasks they would fulfil and the responsibilities they would take before and during the research. Then, those five secondary school Mathematics teachers who accepted these responsibilities prepared, together with the researcher, a more detailed work schedule within the framework of their course schedules.

The teachers who participated in the research were given code names, on the basis of confidentiality: *Teacher A, T, O, M, and S*. Teacher T and Teacher S were working in the school where the implementation was performed, and Teachers O, A, and M were working in other schools. Except for Teacher O, each of the participating teachers were Bachelors of Education; Teacher O was a Bachelor of Science, with a teaching certificate and a Master's Degree in the field of Mathematics (specialising in Algebra). Furthermore, the teaching experience of the teachers involved was Teacher A and Teacher M: 2 years, Teacher S: 5 years, Teacher T: 7 years and Teacher O: 15 years.

The Implementation Process

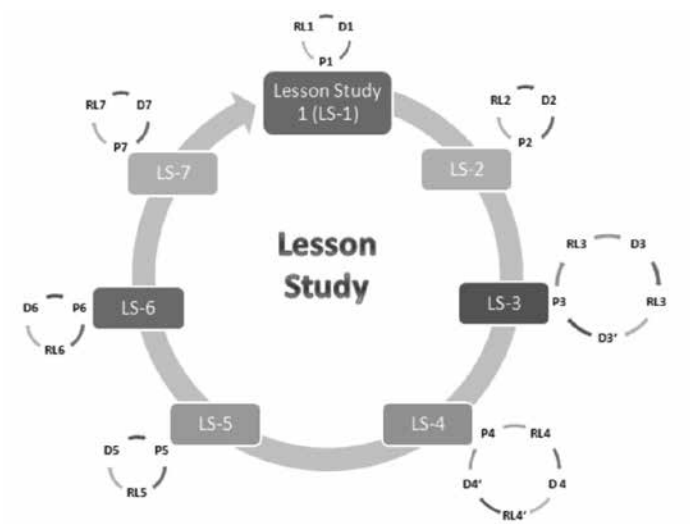
The implementation process of the research took place in two stages: the pilot scheme and the main implementation. In the pilot study (the seminars lasted five weeks and the lesson study was for eight weeks), the activities used in the seminars were tested, the field experts' opinions were taken, and also the researcher who would apply the model for the first time was provided with the experience of it.

Informing the teachers about the main application by visiting schools,

Informing the volunteer teachers by inviting them to the meeting,

Selecting the teachers who would participate in the main research, and conductig the seminars,

Applyng the lesson study (8th Grade - 'Triangles' subject).



Visiting the schools in which the teachers were working an examining their own individual educations

Figure 2. The Main Implementation Process Flow Chart

In the first stage of the main research, the teacher seminars were conducted together with the teachers for a period of five weeks (twice a week for approximately three hours each). In these seminars, the objective of this research, the lesson study model, and the *Geometric Habits of Mind Framework* were explained to the teachers, some discussions were had on the concept of geometric thinking, some geometric problems were solved, and various discussions took place on the teachers' assumptions of how the implementation of the theoretical framework would affect real students by asking them to predict the students' answers to these problems (Figure 2).

In the lesson study sessions, the teachers were asked to teach the lessons covering the objectives involved in the *triangles and the measurement of triangles* sub-learning field of the secondary school 8th-grade geometry learning field in both fall and spring terms. Unlike the pilot study, the lessons were prepared by the teachers individually at the schools in which they work and were also monitored at the end of this process with the aim of assessing the effect of the development in the teachers' geometric thinking on their individual teaching (i.e., on the lessons they were teaching at their own schools, independent from the lesson study). Only the teachers' lesson study processes were included in this research.

The Study Plan and the Organization of the Lesson Studies

In the process of defining the work schedule of the lesson study, we considered the circular structure of the model and the course schedules which the teachers must carry out without any delay in the schools in which they work. At the beginning of the term, the researcher asked the participating teachers to provide the drafts of the course schedules, and, following the interview he had with the school principals, the course schedule of the class was planned according to the days and hours convenient for the teachers in order to come and monitor the research lessons. As per the structure of the model, in case an inefficient research lesson was conducted after the decisions were made by the teachers regarding the planning process, a math class of another group of 8th-grade students in the school was also included within the time frame of the study according to when the teachers were available, with the aim of being able to apply the model with a new class.

During the process, the teachers went every Friday and Monday to the school where the implementation of the model was performed, and they conducted the planning and discussion meetings according to the course schedule over the other days. The teachers met the requirements of the lesson study by coming together on four occasions within the weeks that they felt the need to repeat or continue the research lesson. In accordance with the studies the teachers carried out and the decisions they made, the structures of the 1st, 2nd, 5th, 6th, and 7th lesson studies were applied in three stages: the planning meeting, the research lesson and the discussion meeting. Due to the fact that after the planning meeting, the research lesson, and the discussion meeting, the scheduled time was insufficient for the planned subjects, the teachers felt the need to conduct the research lesson, which is a follow-up of the lesson study, and this lesson was discussed in the 3rd and 4th lesson studies.

Environment

During the research process, the teacher seminars and the lesson study meetings (planning and evaluating) were conducted in a meeting room within the University in which a whiteboard, a computer with various types of geometry software, a projector connected to the computer, tools needed for a geometry lesson (compasses, rulers, protractors, geometry sticks, unit cubes, tangrams, etc.) and other materials (such as paper, scissors, cartons and adhesives) that could be used in the preparation process were made available. We ensured that the teachers could work comfortably as a team in an environment where they could sit face-to-face at a rectangular table.

In the classroom where the research lessons were conducted, two more desks were added at the back to ensure that the natural environment was not disturbed and the teachers were able to observe the in-class implementation of the lesson study. A computer, a projector and the tools needed for a geometry lesson were kept available in the classroom where the lesson took place.

Collecting and Analysing Data

In this research, data were collected by utilizing videotape recordings, the researcher's field notes, teacher-observation forms, a researcher-log, and teacher-logs of the teacher seminars and lesson study sessions (planning, the research lesson, evaluating and putting into a report form) conducted within the context of the research. In this research, data triangulation was employed as the data collection method by utilizing reviews of interviews, observations and documents.

The nature of a lesson study model is that the teachers monitor the research lesson they planned in a real classroom environment together with the whole team for the purpose of analysing it in the discussion meeting. In this research, the teachers were asked to complete the teacher-observation forms at the end of the lesson, which included the components and sub-components of the GHoM prepared by the researcher, with the aim of ensuring that they were able to reflect completely on the geometric thinking in their observations and during the discussion session. In addition, we ensured that professional development was considered by providing each of the teachers with a log and asking them to write down their objectives from the planning and discussion meetings, and noting which of their own aspects of the subject they considered as insufficient. With the data acquired from these, the field notes of the researcher and the researcher-logs were used to verify the data acquired from the videotape recordings.

In the lesson study model, the teachers prepare a report after the lesson is completed including what was done and which things should and should not be done by those who might implement this at a later stage together with their advice. This report is also called a reflective report or final report. The teachers' opinions and revision decisions regarding the lesson plan became more apparent for the researcher with the help of these reports.

In the research, we tried to provide verification of the data gathered from the other resources by making a semi-structured interview with the teacher who taught the lesson after each research lesson. Both sound and imagery recordings of the interviews were made. In these interviews, the following questions were asked of the teachers:

Did the research lesson achieve its objective?
What went well?
What did not go well?
What did not go as planned?
Which geometric habits did you try to promote?
What evidence of which geometric habits did you observe during the lesson process? and
Are there any points which need to be revised?

In this research, the analysis of data was carried out as the "ongoing analysis process" while the implementation was still being performed, and as the "retrospective analysis and the model building process" after the implementation was performed (Steffe and Thompson, 2000; Cobb, 2000; Simon, 2000).

The ongoing analyses together with the individuals' responses form a basis for the unprepared and planned interventions, and also for the interactions which enable the acquisition of additional knowledge, for testing the hypotheses, and for promoting future prospective development. In addition, the knowledge, actions, and characteristics of the individuals were based on developing and changing the model prepared by the researcher (Simon, 2000). However, retrospective analyses continue to develop a descriptor model of mathematical development through re-reviewing the majority of the data and reconsidering all of the related records in a carefully structured way (Simon, 2000). In addition, they guide the creation process of the records in relation to the development of the researchers' conjectures and the development of the individuals' behaviours, opinions, and performances during classroom practice (Molina, Castro and Castro, 2007).

In this research, the analyses of the video-recorded meetings and the research lessons conducted by the teachers during the lesson study process were carried out in seven stages: attentively viewing the video data – describing the video data – identifying critical events – transcribing – coding – constructing a storyline – composing a narrative.

After the analysis of data was carried out, the process was modelled by creating a template for each meeting. With the aim of ensuring the validity and reliability of the analyses of data, data were analysed by two experts in the field of mathematics instruction utilizing the researcher triangulation method, and the external validity of the research was provided in this way.

In the lesson studies applied, with the objective of setting out the development of the teachers' geometric thinking, the data acquired from various data resources, (videos, field notes, teacher-observation notes, teacher-logs, and lesson study reflective reports) during the meetings conducted with the teachers, were analysed using the theoretical framework of *GHoM*, which consists of four geometric habits that have been explained in the theoretical framework, that is, *Reasoning with Relationships*, *Generalizing Geometric Ideas*, *Investigating Invariants*, and *Balancing Exploration and Reflection*.

To ensure that this research was conclusive, long-term interaction with the study group was formed, and triangulation was employed by using different data collection tools. The conclusiveness of the collected data in the discussion meetings' records was scrutinized by checking from the teachers' observation notes whether or not the geometric habits they indicated in the discussion meeting had been defined during the lesson. In the research, we tried to ensure that the data could be transmissible by presenting it through descriptions and direct citations. Data and coding were examined by two Mathematics teachers independent of each other to ensure the research's consistency and confirmation analysis. Moreover, the opinions of field experts were taken about the activities conducted, and all of the implementations performed during the process were video-recorded.

Results

Within the context of the research, the 1st, 2nd, 5th, 6th, and 7th lesson studies covered three stages: Planning (P), The research lesson (RL) and Discussion (D). The 3rd and 4th lesson studies consist of five stages due to the need to conduct a new RL and a new discussion meeting (D) which were, in fact, a follow-up of the research lesson after the stages of planning and discussing the research lesson (Figure 3).

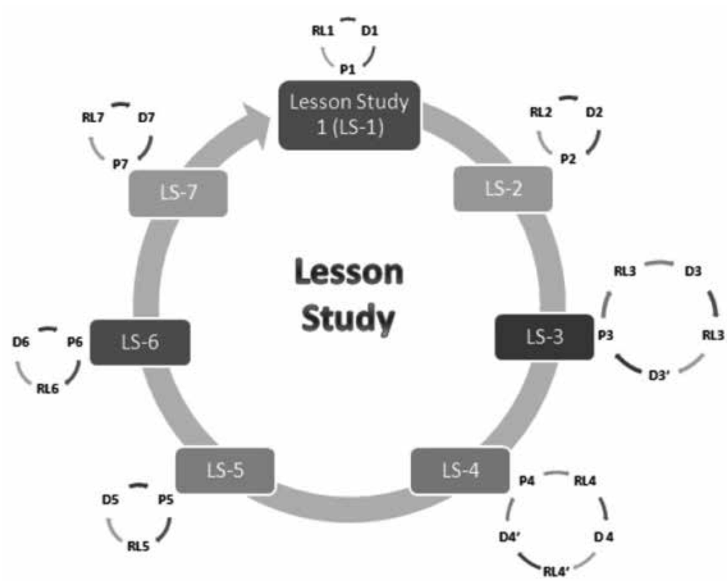


Figure 3. Lesson Study Research Process

Particularly in the first two lesson studies, the teachers indicated that they could not get used to the nature of the method and that they realized when they proceeded

1 **P**: The planning meeting, **RL**: The research lesson, **D'**: Discussion meeting, **RL'**: The follow-up to the research lesson, **D'**: The follow-up to the discussion meeting

to the discussion stage that they had not performed any studies in the planning meetings with the aim of putting right their mistakes, eliminating their deficiencies and promoting their geometric habits. In the 3rd and 4th lesson studies, although the teachers had prepared detailed lesson plans, they could not accomplish those plans within the time period they had projected while applying the RL. In this research, the findings regarding the development of the teachers' geometric habits were based upon two indicators: the teachers' gaps in knowledge about the geometric habits, and the habits which the teachers promoted during the lesson studies.

Table 2

Table of the Teachers' Lacks of Knowledge about the Components

	Reasoning with Relationships		
	Focus on relationships among separate figures	Focus on relationships among the pieces in a single figure	Use spatial reasoning skills
LS -1	---	---	A, T, M, S
LS -2	---	---	A, T, O, M, S
LS -3	---	---	---
LS -4	---	M	S
LS -5	---	---	---
LS -6	---	---	---
LS -7	---	---	---

	Generalizing Geometric Ideas			Investigating Invariants	
	Seek solutions from familiar cases or known solutions	Seek a range of solutions using assumed simplifying conditions	Seek complete solution sets or general rules	Use dynamic thinking and searching	Check evidence of effects
LS -1	A, T, S	---	T, S	A, T	A, S, M
LS -2	A, O, M, S	S	---	O, M, S	---
LS -3	A, T, O, S	S	---	A, T, O, M, S	A, T, O, M, S
LS -4	T, S	T, S	S	---	---
LS -5	A	---	---	---	A
LS -6	---	---	---	---	---
LS -7	---	A, O, S	---	---	---

	Balancing Exploration and Reflection	
	Put exploration in the foreground	Put end goals in the foreground
LS -1	S	---
LS -2	---	---
LS -3	T	A, O, S
LS -4	M, S	---
LS -5	---	---
LS -6	---	---
LS -7	---	---

For instance, in the first lesson study, the fact that Teacher S had a lack of knowledge about the *Checking evidence of effects* component was understood by her expressions given below:

S: *And, hmm, the first triangle the students drew on the notebook, I thought it was checking the evidence of effects…*

R: *Did you think this way because they drew the triangles themselves?*

S: *Yes.*

R: *… You thought this way after which of the triangles were drawn?*

S: *Well, you know, they were trying to draw themselves… I thought about the ones they drew themselves, and what the effect of drawing themselves would be...*

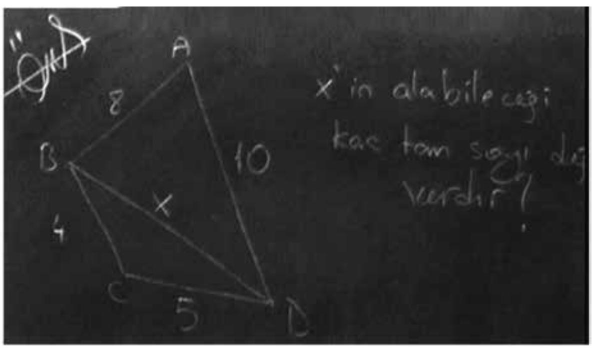
R: *So, you mean, would it be appropriate if it was 3 or 4 [*refers to the side length*]? For example, what must the 3rd side length be?*

S: *Right. And one more thing:* she has drawn triangles with side lengths 7-10-11, 8-9-3...

The following example can be given as an example of the lack of knowledge regarding the fact that although the Using Spatial Reasoning Skills component was promoted, the Symmetry and Proportional Reasoning skills were not used in the same lesson study:

R: *Now I will proceed to another example. In which ways can this question (Problem 4) reveal the geometric thinking?...*

Problem 4



A: *It is only one figure… Focusing on the relationships between the pieces.*

M: *Right, the relationship between the pieces.*

R: *Right, what else?*

A: *She uses spatial reasoning skills.*

R: *What do you mean by "spatial reasoning skill?"*

A: *She separates two of them, then separates two of them [refers to the ABD and BCD triangles], and then puts them together.*

As can be understood from the statements of the teacher, since she mentioned the presence of the *Using Spatial Reasoning Skills* component based on findings which did not refer to this component, this situation showed that she had a lack of knowledge about this subject.

It was included in this research in the context of the habits, of how the teachers finalized the process by promoting which of the geometric habits were used during each lesson study.

Reasoning with Relationships habit of mind

Table 3 shows which components of the *Reasoning with Relationships* habit of mind the teachers tried to promote through the studies they performed at each stage of the lesson study process. In order to identify the teachers' lack of knowledge regarding any of the components in the related lesson studies, the code of the teachers is marked red in the Table below. In addition, their codes are written in bold in order to emphasize if they promoted any of the components in the related lesson studies.

When this Table was analysed, it was observed that regardless of how the teachers tried to promote the *Focusing on relationships among separate figures* and the *Focusing on relationships among the pieces in a single figure* components as in the 1st lesson study, they had difficulties in promoting the *Using Spatial Reasoning Skills* component in comparison to the other components.

Table 3

The Components Regarding the Reasoning with Relationships habit of mind in the Lesson Studies

		Focus on relationships among separate figures	Focus on relationships among the pieces in a single figure	Use spatial reasoning skills
LS-1	P-1	A, T, O, M	A, T, O, M, S	---
	T-1	T, O	T, O, M, S	S
LS-2	P-2	T, O	T, O, M, S	S
	T-2	A, T, O, M	A, T, O, M, S	A, O, M, S
LS-3	P-3	T, O, S	A, T, O, S	O
	T-3	---	A, T, O, S	T, O
	T-3′	O	A, T, O, M, S	A, T, M, S
LS-4	P-4	A, T, O, S	A, T, O, M, S	A, O, S
	T-4	T	T, M, S	O
	T-4′	A, T, O, M, S	A, T, O, M, S	A, T, O, M, S
LS-5	P-5	T, O	A, T, O, M, S	---
	T-5	A, T, O, M	A, T, O, M	A
LS-6	P-6	A, T, O, M	A, T, O, M	A, T, O, M
	T-6	A, T, O, M, S	A, T, O, M, S	A, T, O, M, S
LS-7	P-7	T	A	A, T, O, S
	T-7	A, T, O, M, S	A, T, O, M, S	A, T, O, M, S

In the process of analysing the 1st lesson study's planning meeting videotape records, it was determined that the teachers did not directly include any of the geometric habits. The activities planned by the teachers were analysed in the context of geometric habits and were added into the Table by the researchers. When Table 3 is evaluated considering the teachers' lack of knowledge, it can be observed that the teachers did not have too great a lack of knowledge about the *Reasoning with Relationships* habit of mind after attending the seminars and that they were tending to promote this component intensively. It was observed that the teachers had a lack of knowledge/misconception about the *Using Spatial Reasoning Skills* component only in the first two lesson studies and that they eliminated their deficiencies in the subsequent lesson studies. In addition, it can be observed that as the process proceeded, they tried to promote all of those three components in the studies they conducted.

It can be understood by what the teachers said in the discussion meeting of the lesson study that the activity prepared by the teachers motivated them to highlight reasoning, and therefore to promote the *Reasoning with Relationships* habit of mind, through enabling the students to use the *Proportional Reasoning* skill:

A: *In fact, the* Using Spatial Reasoning Skills *component can also be observed here?*

R: *Are you saying that it can be observed?*

A: *I'm not sure.*

M: *I have told you that it can be observed.*

A: *I just realized it; yes, it does seem like it can be observed.*

R: *For example, which way of reasoning are you talking about when talking about the "spatial reasoning?"*

A: *I got that feeling, I mean, as if it was available there; I mean he said it was available.*

O: *Spatial reasoning skills?*

A: *I mean he said it was available there.*

O: *The "spatial reasoning" is... through making reasoning proportionally between two or among more than two geometric figures... He will make proportional reasoning between the two of them... The side of one of the triangles is 1,5 times more than the side of the other triangles.*

As can be seen from these comments of the teachers, both the processes of analysing in relation to geometric thinking and intellectualising the geometric habits in the discussion meeting developed the *Reasoning with Relationships* habit of mind through using spatial reasoning skills.

Generalizing Geometric Ideas habit of mind

The components which the teachers promoted regarding the *Generalizing Geometric Ideas* habit of mind in the studies they performed during the lesson study process are given in Table 4. When the teachers' gaps in knowledge regarding the *Generalizing Geometric Ideas* habit of mind were taken into consideration, it could be seen that they had too many gaps in knowledge, particularly about the *Using Familiar Cases* component in the first three lesson studies. When the other components about the *Generalizing Geometric Ideas* habit of mind were examined, it was observed that the teachers promoted these components more in the 5th, 6th, and 7th lesson studies.

Following a discussion about the activity's outline, Teacher O and Teacher A had another discussion about whether or not they were promoting the situations which will encourage the students to generalize using the secondary elements of a triangle:

***A**: For example, you will give an ordinary triangle, right? In other words, four of them… I mean, as far as I understand, its height is over this one... as far as our friend said... that is to say, its median is over this one... So, that's what I'm confused about: he must be able to construct a median also in 3-4 triangles. If he can construct it in a scalene triangle, he must be able to construct it both in an isosceles triangle and an equilateral triangle...*

When the *Generalizing Geometric Ideas* habit of mind was examined in the context of *Using Familiar Cases*, it was observed that the teachers tried to promote this component from the 1st lesson study. However, along with the researcher's guidance, due to the fact that the teachers had too great a lack of knowledge about this component in the first three lesson studies, it was observed that these deficiencies were starting to be eliminated as of the 5th lesson study. It was observed that the teachers were not lacking in knowledge about the *Using Assumed Simplifying Conditions* which is another component of the *Generalizing Geometric Ideas* habit of mind and also did not try to promote this component too much until the 5th lesson study. Moreover, it was recognised that some of the teachers had a lack of knowledge about this component when they proceeded to the 7th lesson study.

Table 4

The Components Regarding the Generalizing habit of mind in the Lesson Studies

		Seek solutions from familiar cases or known solutions	Use Assumed Simplifying Conditions	Seek complete solution sets or general rules
LS-1	P-1	A, T, O	A, T	A, T, O
	T-1	T, O, S	O	T, O
LS -2	P-2	T, O, S	O	T, O
	T-2	A, T, O, M, S	O, M	A, T, O, M, S
LS -3	P-3	A, T	---	---
	T-3	T, O, S	A, T, O, S	T, O, S
	T-3′	A, T, O, M, S	O, S	A, T, O, S
LS -4	P-4	A, T, O, S	S	---
	T-4	T, M	---	---
	T-4′	A, T, O, M, S	A, T, O, S	T, S
LS -5	P-5	---	A, T, O, M, S	---
	T-5	A, T, O, M	---	A, T, O, M
LS -6	P-6	A, T, O, M	O	O, M
	T-6	A, T, O, M, S	T	A, T, O, M, S
LS -7	P-7	T	A, T, S	A, T, S
	T-7	A, T, O, M, S	A, T, O, M, S	T, S

In the context of promoting the *Seeking complete solution sets or general rules* component, when the lesson studies conducted by the teachers were examined, it was observed that all of the teachers in the 2nd and 6th lesson studies and almost all of the teachers in the 3rd and 5th lesson studies tried to promote this component. In addition, it was observed that except for Teacher T (in the 1st lesson study) and Teacher S (in the 4th lesson study), none of the teachers had a lack of knowledge about this component.

Investigating Invariants habit of mind

The components which the teachers promoted regarding the *Investigating Invariants* habit of mind at each stage of the lesson study process are given in Table 5. In order to indicate the teachers' lack of knowledge regarding any of the components in the related lesson studies, the code of the teachers is marked in red in the Table. When Table 5 was reviewed in general terms, it could be observed that the teachers tried to promote all of the components as of the 1st lesson study. When the teachers' studies were examined regarding promoting the *Investigating Invariants* habit of mind, it was observed that all of the teachers had too great a lack of knowledge about this component in the first three lesson studies. It was also observed that there was a decrease in their lack of knowledge and they promoted both components as of the 4th lesson study.

When the teachers' lack of knowledge about the components of this habit was evaluated together with the studies they conducted for promoting them, it was possible to say that they tried to eliminate their lack of knowledge about the *Investigating Invariants* habit of mind as of the 4th lesson study and to promote this habit until the end of the research.

Table 5

The Components Regarding the Investigating Invariants habit of mind in the Lesson Studies

		Use dynamic thinking and searching	Check evidence of effects
LS-1	P-1	A, T, O, S	T, O, S
	T-1	S	---
LS -2	P-2	S	---
	T-2	A, T, O, M, S	A, O, M, S
LS - 3	P-3	---	----
	T-3	A	T
	T-3'	A, T, O, M, S	A, T, O, M, S
LS - 4	P-4	---	---
	T-4	T, O	---
	T-4'	A, T, O, M, S	A, T, O, M, S
LS -5	P-5	T, O, S	---
	T-5	A, T, M	A, M
LS -6	P-6	---	A, T, O, M
	T-6	O, M	---
LS -7	P-7	A, M, S	A, T, M, S
	T-7	A, T, O, M, S	A, T, O, M, S

Moreover, it was observed that Teacher S had a lack of knowledge about the *Checking evidence of effects* step of the *Investigating Invariants* component.

It can be understood by the comments of Teacher S below that she confused this step with the Exploration step of the *Balancing Exploration and Reflection* component which is another geometric habit:

S: *And, hmm, in the first triangle the students drew on the notebook, I thought it was, I mean, checking evidence of effects when investigating invariants … And also...*

R: *Did you think this way because they drew the triangles themselves?*

S: *Yes.*
R: *You thought this way after which of the triangles were drawn?*
S: *Well, you know, they were trying to draw themselves… I thought about the ones they drew themselves, and what the effect of drawing themselves would be...*
R: *So, you mean, would it be appropriate if it was 3 or 4? Was it like, what must the 3rd side length be, for instance?*
S: *Right.*

Balancing Exploration and Reflection habit of mind

The components which the teachers promoted regarding the *Balancing Exploration and Reflection* habit of mind in the studies they performed during the lesson study process are given in Table 6. With the aim of indicating the teachers' lack of knowledge regarding any of the components in the related lesson studies, the code of the teachers is marked in red in the Table.

It was observed that the teachers were tending to promote the *Putting exploration in the foreground* component in the first three lesson studies, and the *Putting end goals in the foreground* component in the last three lesson studies. It was observed that, although they attached importance to the *Putting end goals in the foreground* component, they could not understand the *Putting exploration in the foreground* component sufficiently.

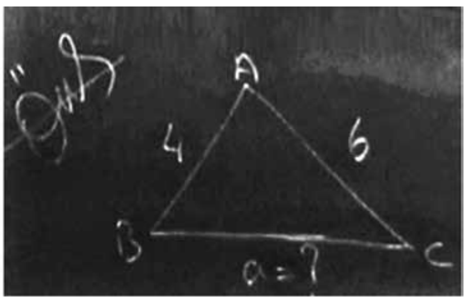


Figure 5. The Given Problem in the Research Lesson 3

It was observed that the teachers used the problem given above (Figure 5) ("In this triangle, what can the side length of A be?") at the beginning of the lesson study process they conducted. It can be understood by the following comments of the teachers that the students used this problem to make verifications on the subject of triangle inequality through examples of "constructing-not constructing a triangle," and that the teachers confused the *Balancing Exploration and Reflection* component's *Putting exploration in the foreground* step with the *Checking evidence of effects* step:

R: *Here* [Figure 5], *we asked which values can be assigned to A. Is there anything in your notes you would like to add about this? What about you, Teacher M? Which components did we use here about geometric thinking?*
M: *Probably the "Checking evidence of effects", I guess...*

R: *Is that possible? If so, for example, in which ways can this be?*
M: *Once again they are trying to use the relation within the examples.*

Table 6
The Components Regarding the Balancing Exploration and Reflection habit of mind in the Lesson Studies

		Put exploration in the foreground	Put end goals in the foreground
LS -1	P-1	A, T, O, M, S	A, T, O
	T-1	T, O, S	---
LS -2	P-2	T, O, S	---
	T-2	A, O, M	A, T, O, M, S
LS -3	P-3	A, T, O	---
	T-3	T	T, O
	T-3'	A, T, O, M, S	A, O, S
LS -4	P-4	---	T
	T-4	T, O, S	T, O, S
	T-4'	A, T, M, S	M, S
LS -5	P-5	T, O, S	A, T, O, S
	T-5	A, T, M	A, T, O, M, S
LS -6	P-6	T	A, T, O, M
	T-6	---	A, T, O, M, S
LS -7	P-7	A, T, S	A, M, S
	T-7	O	A, T, O, M, S

When the evidence regarding the teachers' *Balancing Exploration and Reflection* habit of mind was reviewed, it was observed that although they had a lack of knowledge until the 5th stage, they tried to promote this habit from the 1st stage.

The development in the teachers' geometric thinking is reflected in their decreased lack of knowledge and also in their comments about this component, particularly as of the 5th lesson study. For example, Teacher T explained the difference between the *Putting end goals in the foreground* and the *Putting exploration in the foreground* components through a metaphor:

"Now let's imagine that a vessel sank and saw the bottom of the sea. Making a research study into the vessel is 'making an exploration', but diving again for something you have just explored there is 'your purpose'. When we say that there is a vessel at the bottom of the sea, they directly start to search for the gold at the bottom of the sea."

Discussion and Conclusions

The *Geometric Habits of Mind* (Driscoll et al., 2007) identifies and explains the ways of thinking (*Reasoning with relationships, Generalizing Geometric Ideas, Investigating Invariants habit of mind and Balancing Exploration and Reflection habit of mind*) which both students and teachers should utilize in order to be successful geometric problem-solvers. In fact, it is not an easy process to identify whether a teacher is a good geometric problem-solver or not.

In the process of planning a lesson study, the teachers should predict the students' possible answers and take those details into consideration in the planning of the lesson

(Murata, 2011). In this research, in the first two lesson studies, the teachers indicated that they could not get used to the nature of the method and that they realized after proceeding to the discussion stage that they had not performed any studies in the planning meetings for correcting their mistakes, for eliminating their deficiencies, and for promoting their geometric habits. In the 3rd and 4th lesson studies, the teachers prepared detailed lesson plans, however, they could not accomplish those plans within the time they had projected while conducting the RL. This situation might have arisen from the fact that the RL was not constructed in a detailed way in the planning process and by putting the students' opinions in the foreground, as Murata (2011) has emphasized.

It was observed that the teachers had a lack of knowledge about many of the habits in the first four lesson studies, researcher interventions were made with the aim of eliminating the deficiencies, and within the process, there was a decrease in the teachers' lack of knowledge about their habits, especially during the studies following the 4th cycle. When the teachers' lack of knowledge regarding this was examined, it was observed that the 2nd lesson study for *Reasoning with relationships*, the 4th lesson study for *Generalizing Geometric Ideas*, the 3rd lesson study for the *Investigating Invariants habit of mind,* and the *Balancing Exploration and Reflection habit of mind* were milestones. In this case, considering that the milestone for the lesson studies conducted for the research was, in fact, the 5th lesson study, it is suggested the training sessions which will be planned will need to be carried on for at least a period of five weeks. In addition, the teachers who receive this training regarding conducting lesson studies in their own schools will guide the teachers who are trying to achieve the lesson study model in their schools.

It was concluded that the teachers tried to promote all of the components regarding the *Reasoning with Relationships habit of mind* while both planning and conducting the lesson, and in discussing the lesson they have taught. However, the fact that the teachers had difficulties in promoting the *Using Spatial Reasoning Skills* component while trying to promote the *Focusing on relationships among separate figures* and the *Focusing on relationships among the pieces in a single figure* components as of the 1st lesson study, can be explained by the nature of the component. Since this component is closely associated with utilizing proportional reasoning in geometry and symmetry, it is possible to relate this to the fact that the subject matter is not appropriate for using these concepts.

When the Table was analysed, it was observed that although the teachers tried to promote the *Focusing on relationships among separate figures* and the *Focusing on relationships among the pieces in a single figure* components from the 1st lesson study, they had difficulties in promoting the *Using Spatial Reasoning Skills* component in comparison to the other components.

It is possible to say that although the teachers had too great a lack of knowledge about the *Using Familiar Cases* component of the *Generalizing Geometric Ideas habit of mind,* they tried to promote the *Generalizing Geometric Ideas* through its other components. This situation can be considered as a sign that the evidence collected for the *Generalizing Geometric Ideas habit of mind* has shown it to be hierarchical by Driscoll et al. (2007).

In *Investigating Invariants habit of mind,* it was observed that although the teachers could not promote both ways of reasoning during the first three lesson studies, they started to promote them as of the 4th lesson study. This situation might have arisen from the fact that the subjects instructed in the first three lessons were "constructing the main and secondary elements of a triangle". Thus, Erduran and Yeşildere (2010) have determined in their research about "geometric structure building of teachers using compasses and straight sides" that the lessons are taught in a teacher-centred way, and the students try to follow the teacher's instructions through rote learning. This situation might have caused the teachers to design lessons richer in content in terms of geometric thinking as they proceeded to the next topics, and to promote the geometric habits of *Dynamic Thinking and Searching* and *Checking evidence of effects*.

Regarding the *Balancing Exploration and Reflection habit of mind*, which is the last *Geometric Habit of Mind*, it was observed that the teachers did not lack too much knowledge. It is possible to say that the teachers did not have many discussions about this, since it is difficult to promote this habit, and therefore their lack of knowledge might not be fully revealed. It is necessary for the teachers to create problems which will force the students to develop their geometric habits in the problem-solving process by putting exploration in the foreground or putting end goals in the foreground. The fact that this situation started to emerge from the 5th lesson study onward can be explained by the development which also occurred in the teachers' knowledge of the field. When it is taken into consideration that the teachers' knowledge of geometry has a significant effect on the students' geometry learning, the above-mentioned situation plays a critical role (Lenhart, 2010; Clements, 2003).

The lesson study model's effect is obvious when one takes into account the fact that, in order to develop the teachers' geometric thinking, it is also necessary to develop their Geometric Habits of Mind. With the aim of developing the teachers' ways of reasoning about geometric thinking, it is necessary to confront them with challenging geometry problems, to make them predict the students' possible answers about these problems, and to ensure that they have discussions with their colleagues about those answers during their studies. Therefore, it is suggested that the lesson study model is made more widespread, primarily for the purpose of developing the teachers' geometry field knowledge and geometric thinking. And in order to provide this, the seminars teaching this model must be included in the in-service training programs of the countries. In this way, it can be ensured that the teachers in the first instance and the next generations immediately following can be better problem-solvers in terms of geometry.

Acknowledgement

This study is a part of the doctoral dissertation of Deniz Özen under the supervision of Assoc. Prof. Dr Nilüfer Köse.

This study was supported by Anadolu University Scientific Research Projects Commission under the grant no: 1308E325.

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Studija predavanja u svrhu razvoja geometrijskih mentalnih navika nastavnika

Sažetak

Glavni cilj ove studije je proučavanje razvoja geometrijskoga razmišljanja učitelja matematike u osnovnim školama. Kako bi se to postiglo, model istraživanja definiran je kao studija predavanja. Proces provedbe studije proveden je uz sudjelovanje pet nastavnika matematike iz različitih škola. Prvo je s nastavnicima održano pet tjedana seminara. Kao dio ovoga procesa, studija predavanja objašnjena je teorijskim okvirom geometrijskih mentalnih navika te je isti primijenjen u praksi. Zatim je izvedena studija predavanja koja je trajala oko tri mjeseca. Prikupljanje podataka provedeno je pomoću nekoliko alata za prikupljanje podataka, a analiza podataka izvedena je pomoću modela videoanalize koju su razvili Powell, Francisco i Maher (2003). Kao rezultat ovoga istraživanja, geometrijsko razmišljanje nastavnika poboljšalo se tijekom studije predavanja. U procesu od prve do sedme studije predavanja, primijetili smo poboljšanje matematičkoga jezika kojim se koriste nastavnici, njihova objašnjenja, pitanja učenika u učionici, kao i aktivnosti i zadatke koje su nastavnici stvorili na temelju geometrijskih mentalnih navika za relevantne pojmove. U procesima učenja, promatrali smo i nastavnike kako razmatraju i vrednuju ove komponente te ih uključuju u planiranje i provedbu svojih satova geometrije. Nakon studija predavanja, nastavnici su također razmatrali geometrijske navike u vlastitim školskim okruženjima i razmišljali o procesu učenja putem pripremljenih aktivnosti i problema koji su se tad definirali.

Ključne riječi*: geometrijske mentalne navike; geometrijsko razmišljanje; profesionalno usavršavanje nastavnika; studija predavanja*

Uvod

Geometrija može omogućiti pojedincima da analiziraju i rješavaju probleme te da uspostavljaju veze između matematike i života, zbog toga što pruža pojedincima mogućnost tumačenja fizičkoga okruženja. Geometrija se također koristi kao alat za proučavanje znanosti i drugih srodnih tema (Clements i Battista, 1992). Pojedinci bi trebali imati određenu razinu znanja geometrije kako bi, na primjer, mogli izvesti geometrijska mjerenja ili služiti se vještinama geometrijskoga razmišljanja, osobito kada se suočavaju sa stvarnim životnim izazovima. Ove će vještine učiniti njihov svakodnevni život lakšim. Geometrijsko razmišljanje definirano je kao „sposobnost razmišljanja o geometrijskim situacijama i donošenja zaključaka" (Van de Walle, 2004). Na temelju ove definicije, može se reći da se geometrijsko razmišljanje razvija kroz geometrijsko rasuđivanje i geometrijske odnose, a predstavlja i sposobnost donošenja zaključaka na temelju tih generalizacija.

Učenje geometrije je proces koji počinje tijekom predškolskoga odgoja i nastavlja se tijekom srednjoškolskoga obrazovanja. U osnovnoj školi učeničko znanje geometrije koje je već intuitivno mora biti konceptualizirano i razvijeno pomoću opipljivih modela (Ersoy, 2006). Na taj način djeca mogu primijeniti svoje geometrijsko znanje na probleme s kojima se suočavaju u stvarnom životu. Iako je primarna svrha učenja geometrije definirana kao „učenikova sposobnost primjene geometrije u procesu rješavanja problema i objašnjavanja vlastitog fizičkog svijeta, okoliša i svemira", euklidska geometrija koja se koristi u postojećim matematičkim programima nije dovoljna da pomogne učenicima razumjeti svoje okruženje i uspostaviti odnose s okolinom u kojoj žive (Baki, 2001). Stoga je istraživanje procesa poučavanja neophodno kako bi učenicima pružilo uvid u geometrijske pojmove i omogućilo im da pronađu smisao u njima (Clements i Battista, 1992). Cilj je obrazovanja prenijeti učenicima geometrijska znanja i vještine. Ovo ne bi smjelo biti ograničeno na jednostavno pružanje geometrijskoga znanja, već bi također trebalo osigurati da učenici usvoje vještine geometrijskoga rasuđivanja i sposobnost geometrijskoga razmišljanja. Stoga bi cilj nastavnika trebao biti razvoj geometrijskoga razmišljanja uz stjecanje znanja i vještina uključenih u program.

Kako bi učitelji matematike mogli pružiti dobro geometrijsko obrazovanje, moraju imati duboko razumijevanje geometrije (Jones, 2000). Istraživanja pokazuju da sadašnji i budući učitelji matematike nemaju dovoljno znanja o geometriji (Fuys, Geddes i Tischler, 1988; Swafford, Jones i Thornton, 1997; Mayberry, 1983); pogreške koje su napravili učitelji matematike pokazuju paralele s pogreškama koje su napravili učenici (Herskowitz i Vinner, 1984; Swafford, Jones i Thornton, 1997), a učiteljsko poznavanje geometrije ima značajan utjecaj na poučavanje učenika (Lenhart, 2010; Clements, 2003). Ako učitelji dobro razumiju geometriju, moći će bolje razumjeti poteškoće s kojima se učenici suočavaju. Stoga je moguće tražiti rješenja za probleme koji se javljaju tijekom procesa poučavanja (Durmuş, Toluk i Olkun, 2002). Da bi se razvijalo učiteljsko znanje geometrije i njihovo poučavanje geometrijskih sadržaja, moraju imati dovoljno znanja i iskustva (Toluk, Olkun i Durmuş, 2002). Iako je istaknuto da učinkovito okruženje za učenje mogu pružiti samo iskusni učitelji (Putnam, Heaton, Prawat i Remillard, 1992), također je potrebno uzeti u obzir i učiteljevo znanje sttečeno tijekom njegova stručnog osposobljavanja. Naime, primjeri iz prakse pokazuju na mnogi učitelji nemaju nikakvu podlogu u području geometrije i stoga je njihovo poučavanje geometrije neučinkovito.

Kako se učiteljsko znanje geometrije i svijest učenika o vlastitim kognitivnim procesima poboljšavaju, uočljive su i promjene u sadržaju i načinu poučavanja (Swafford, Jones i Thornton, 1997; Mistretta, 2000). Kada se razmatraju rezultati dobiveni na razini znanja učitelja, dolazi se do zaključka da i nastavnici trebaju dobro znati geometriju (toluk, Olkun...) kako bi je mogli i uspješno poučavati uzimajući u obzir (Toluk, Olkun i Durmuş, 2002). Iako učitelji moraju imati praktično znanje (Tanışlı, 2013), istraživači naglašavaju da dovoljno znanja ne znači da će učitelji matematike biti uspješni u poučavanju te da bi pojedinci trebali sintetizirati svoje matematičko znanje i znanje o poučavanju (Kahan, Cooper i Bethea, 2003; Türnüklü, 2005). Rezultati dobiveni tijekom nedavno provedenih istraživanja i primjeri dobre prakse pokazuju da kada učenici posjeduju iscrpno znanje o geometriji i razvijeno geometrijsko rasuđivanje, u stanju su prevladati poteškoće s kojima se suočavaju u učenju drugih područja matematike. Učitelji bi stoga trebali shvatiti važnost geometrije unutar matematičkoga programa, a istraživači trebaju pružiti tim učiteljima znanje i alate koji će im pomoći da razviju svoje vlastite aktivnosti (Gutiérrez, 2014). Američko nacionalno vijeće učitelja matematike – NCTM također naglašava da učitelji moraju pomoći svojim učenicima u promicanju geometrijskoga rasuđivanja i proširiti njihovo znanje o konceptima, uz proceduralno znanje (NCTM, 2000). Svrha ovoga istraživanja je primjena modela studije predavanja u kontekstu razvoja geometrijskoga razmišljanja kod učitelja. Ovo se temelji na ideji da učitelji imaju ključnu ulogu u razvoju poučavanja geometrije i geometrijskoga razmišljanja.

Teorijski okvir

Pri razmatranju istraživanja razvoja geometrijskoga razmišljanja pojedinaca, može se primijetiti da postoje dvije važne teorije koje su predložene uglavnom u Piaget i sur. (Piaget i Inhelder, 1956; Piaget, Inhelder i Szeminska, 1964) i van Hiele (1986). Iako istraživanje geometrije ima dugu povijest, prva poznata studija o razvoju geometrijskoga razmišljanja objavljena je u knjizi *Dječja ideja prostora* koju su napisali Piaget i Ingelder (1956.) 1948. godine. Iako ovo istraživanje već godinama privlači značajnu pozornost, posljednjih je godina suočeno s velikim kritikama u smislu bavljenja pretpostavkama matematičkih definicija i razmatranja poboljšanja geometrijskoga razmišljanja kod djece (Bjorklund, 1997; Clements i Battista, 1992; Darke, 1982; Geeslin i Shar, 1979; Kapadia, 1974; Laurendau i Pinard, 1970; Martin, 1976; Peel, 1959; Somerville i Bryant, 1985; van Der Sandt, 2000).

Nakon istraživanja Piageta i Ingeldera (1956), nizozemski odgojiteljski stručnjaci Pierre Marie van Hiele i Dina van Hiele-Geldof proveli su istraživanje u kojem je značajna pozornost usmjerena na geometrijsko razmišljanje. Ova je studija ispitivala pitanja formiranja geometrijskih ideja i razvoja geometrijskoga razmišljanja kod djece. Razvrstavanjem kroz razine (vizualizacija/prepoznavanje, analiza, neformalna dedukcija/poredak, formalna dedukcija te strogost) usustavljene su faze razvoja geometrijskoga razmišljanja temeljene na sposobnostima pojedinaca, počevši od razine razvrstavanja oblika prema njihovoj sličnosti, razlikama i izgledu, sve do razine povezanosti različitih sustava s prostorima (Van Hiele, 1986). Prema ovom modelu, van Hieleov test koristi se za otkrivanje geometrijskoga razmišljanja kod pojedinaca. Neki istraživači (Manizade, 2006; Lenhart, 2010) ističu da, iako se različite verzije van Hieleova testa koriste za pouzdanu procjenu učiteljskoga znanja geometrije (Usiskin i Senk, 1990; Wilson, 1990), ovaj test ne procjenjuje njihovo pedagoško znanje. Stoga je u procesu razvoja geometrijskoga razmišljanja učitelja i analize ovog razvoja, u ovoj studiji korištene su *geometrijske mentalne navike – GHoM (Geometric Habits of Mind)*, koje predstavljaju potpuno novi teorijski okvir u usporedbi s gore navedenim pristupima.

Struktura *geometrijskih mentalnih navika* koju su predložili Driscoll, Wing DiMatteo, Nikula i Egan (2007) i koja ima značajnu ulogu u razvoju geometrijskoga razmišljanja jest model koji je razvijen od strukture *matematičkih mentalnih navika – MHoM (Mathematical Habits of Mind)* (Cuoco, Goldenberg i Mark, 1996). *Matematičke mentalne navike* definirane su kao posebni načini razmišljanja koji su slični tehnikama koje matematičari koriste u svojim pristupima rješavanju problema i razmišljanju o matematičkim konceptima (Cuoco, Goldenberg i Mark, 1996; 2010). Osim toga, *matematičke mentalne navike* također su definirane kao učinkoviti načini razmišljanja koji promiču učenje, primjenu i razumijevanje formalne matematike (Driscoll i sur., 2007). Goldenberg, Cuoco i Mark (1998) istaknuli su da se matematička snaga može savršeno definirati skupom matematičkih navika. Ovaj koncept je načelo organizacije koju su predložili Cuoco, Goldenberg i Mark (1996) za program matematike u kojem učenici srednjih škola i sveučilišni studenti razmišljaju o metodama kojima se koriste matematičari. Predložene su *matematičke mentalne navike* kako bi se studentima omogućilo razumijevanje načina razmišljanja matematičara (Lim i Selden, 2009). Svrha ovih navika definirana je kao „pomaganje studentima da nauče i usvoje načine razmišljanja matematičara o problemima"(Cuoco, Goldenberg i Mark, 1996; Lim i Selden, 2009).

Geometrijske mentalne navike (GHoM) razvijene su u kontekstu razvoja projekta geometrijskoga razmišljanja temeljenoga na prethodnim studijama koje su proveli Driscoll i sur. vezane za *matematičke mentalne navike*. To je okvir koji naglašava učinkovite načine razmišljanja koji se posebice temelje na geometrijskom razmišljanju. U usporedbi s okvirom *matematičkih mentalnih navika*, ova struktura uključuje komponente sukladne geometrijskom razmišljanju. Istraživači koji su radili na ovoj strukturi četiriju međusobno povezanih *geometrijskih navika* (Driscoll, Wing DiMatteo, Nikula, Egan, Mark i Kelemanik, 2008) pokazali su da se ona temelji na perspektivi geometrijskoga razmišljanja. Teorijski okvir *geometrijskih mentalnih navika* pomaže nastavnicima da vide jesu li učenici razmišljali u procesu rješavanja geometrijskih problema, zahvaljujući sredstvima kojima uključuju učinkovite načine rješavanja geometrijskih problema. Ovaj se okvir usredotočuje na određivanje dokaza geometrijskoga razmišljanja. Unutar teorijskoga okvira identificirani su načini za učenike i odrasle kako bi postali uspješni u rješavanju geometrijskih zadataka, a uključena je i analiza dokaza geometrijskoga razmišljanja (Köse i Tanışlı, 2014).

Četiri glavne komponente *geometrijskih mentalnih navika* definirane su kao *rasuđivanje o odnosima, generalizacija geometrijskih ideja, istraživanje invarijanti* te *balansiranje istraživanja i razmišljanja* (Driscoll i sur., 2008).

Mentalna navika *rasuđivanja o odnosima* može se definirati kao „aktivno traženje odnosa unutar geometrijskih oblika i između geometrijskih oblika (poput pratnje ili sličnosti)" (Driscoll i sur., 2008). Ovi odnosi mogu biti između zasebnih geometrijskih oblika, između dijelova jednog oblika, u samome obliku u cjelini ili između pojmova (kao što su površina i opseg). Pitanja koja otkrivaju ovu naviku definirana su kao: *Jesu li ti oblici slični? Na koji način?*, *Na koliko će načina biti slični?*, *Jesu li ovi oblici različiti? Na koji način?*, *Što sam trebao učiniti kako bi ovaj oblik izgledao poput onog drugog*? (Driscoll i sur., 2008). Ova se navika usredotočuje na odnos između različitih oblika, jednog oblika ili elemenata unutar jednog oblika, a uključuje i neke vještine razmišljanja u kojima se rabi proporcija i simetrija.

Mentalna navika *generalizacije geometrijskih ideja* može se definirati kao „želja za razumijevanjem i opisivanjem izraza „*često*" i „*uvijek*", koji su povezani s geometrijskim konceptima i operacijama" (Driscoll i sur., 2008). Mentalna navika *generalizacije geometrijskih ideja* postiže se kroz faze izgradnje hipoteza o izrazima *obično*, *svaki* i *u koliko slučajeva*; testiranja hipoteza; donošenja zaključaka o hipotezama; pružanje snažnih argumenata koji će potkrijepiti rezultat. Pitanja koja otkrivaju ovu naviku mogu biti, npr. *Događa li se ovo u svakom slučaju?*, *Zašto bi se ovo dogodilo u svakom slučaju?*, *Jesam li pronašao sve koji se uklapaju u ovaj opis?*, *Mogu li dati primjere kad ovo nije točno, a ako je tako, trebam li ponovno razmotriti svoju generalizaciju?*, *Zašto je ovo primjenjivo u drugim dimenzijama?* (Driscoll i sur., 2008).

Mentalna navika *istraživanja invarijanti* može se definirati kao „analiza o tome na koja svojstva geometrijskog oblika utječe transformacija (npr. zrcaljenje, paralelni pomak, rotacija, itd.)" (Driscoll i sur., 2008). Ovdje su invarijante definirane kao „svojstva koja uvijek ostaju ista, čak i ako se druga stanja oblika mijenjaju." Nepromijenjena svojstva oblika u transformaciji su njegova površina, opseg, volumen, duljina bridova/stranica, kao i omjer duljine bridova/stranica i kutova. Pitanja koja otkrivaju ovu naviku definirana su kao: *Kako ovo možemo premjestiti s one pozicije na ovu?*, *Što se promijenilo? Zašto?*, *Što se nije promijenilo? Zašto?* (Driscoll i sur., 2008). Istraživanja ove navike pokazala su da kad učenici promišljaju o komponenti *istraživanja invarijanti*, oni razumiju situacije kad obraćaju pozornost na učinak transformacije.

Tablica 1

Mentalna navika balansiranja istraživanja i razmišljanja može se definirati kao „razmišljanje o pokušaju usvajanja različitih pristupa koji su obično odabrani kao rezultat predloženih hipoteza i redovito uzimanje u obzir onoga što je dosad naučeno" (Driscoll i sur., 2008). Važno je, koliko god je to moguće, zadržati ravnotežu između istraživanja koja podupiru hipoteze i razmišljanja o onome što je naučeno kao rezultat tih istraživanja. Pitanja koja otkrivaju ovu naviku mogu biti, npr. *Što će se dogoditi ako* *nacrtam sliku i/ili ako dodam ili uklonim ovu sliku, i/ili ako počnem razmišljati iz gledišta posljedica i u suprotnom smjeru, itd.?*, *Što mi taj postupak znači?*, *Kako moji prvi pokušaji rješavanja ovog problema mogu objasniti trenutni pristup koji poduzimam?* (Driscoll i sur., 2008).

Ukratko, *rasuđivanje o odnosima* je razmišljanje o geometrijskim oblicima kroz istraživanje geometrijskih odnosa i korištenje posebnih vještina rasuđivanja; *generaliziranje geometrijskih ideja* je razumijevanje i identifikacija geometrijskih činjenica; *istraživanje invarijanti* je istraživanje promjenjivih i nepromjenjivih stanja i svojstava u geometrijskoj strukturi, a *balansiranje istraživanja i razmišljanja* je pokušaj korištenja različitih načina rješavanja problema i povratka na prve korake kako bi se proces redovito revidirao. Ove navike i njihove komponente definirane su u tablici 1 na način koji je korišten u istraživanju koji su proveli Driscoll i sur. (2008). Osim istraživanja koje su proveli Driscoll i sur. (2007), postoje još dvije studije u nacionalnoj literaturi povezane s tom temom. To je istraživanje koje je proveo Bülbül (2016) a kojim se procijenilo znanje budućih učitelja matematike, studija predavanja koju su proveli Özen i Köse (2013) o geometrijskim objektima koja je primijenjena na učitelje matematike te istraživanje koje su proveli Köse i Tanışlı (2014) koje je uspostavilo geometrijske navike budućih osnovnoškolskih učitelja.

Cilj ovoga istraživanja bio je proučavanje razvoja geometrijskoga razmišljanja kod srednjoškolskih nastavnika matematike primjenom studije predavanja. U skladu s tim ciljem, ovo je istraživanje osmišljeno kako bi odgovorilo na pitanje *kako se geometrijsko razmišljanje srednjoškolskih nastavnika matematike razvija kao dio GHoM-a u procesu provedbe modela studije predavanja?*

Metodologija

Pri razmatranju studija stručnoga usavršavanja nastavnika može se primijetiti da *studija predavanja* zauzima svoje mjesto među popularnim pristupima koji se posebice primjenjuju za stručno usavršavanje nastavnika matematike (Stigler i Hiebert, 1999; Lewis, Perry, Hurd i O' Connell, 2006; Watanebe, 2002; Fernandez, 2002; Saito, 2012; Lewis, Perry, Friedkin i Roth, 2012). Ovo je model stručnoga usavršavanja koji omogućuje sudionicima uvid u primjene u učionici iz druge perspektive (Stigler i Hiebert, 1999), a također je definiran kao „suradnički pristup stručnom usavršavanju za učitelje" (Murata, 2011; Lewis i Tsuchida, 1998; Stigler i Hiebert, 1999). Ovaj pristup uključuje etape u kojima se učitelji okupljaju te zatim zajednički planiraju, primjenjuju i procjenjuju učinkovito predavanje koje će omogućiti učenicima da nauče potrebne sadržaje (Murata, 2011; Baki, Baki i Arslan, 2011). U ovome je istraživanju upotrijebljen model *studije predavanja* kao kvalitativna metoda istraživanja s ciljem razvijanja geometrijskoga razmišljanja srednjoškolskih nastavnika u vlastitom školskom okruženju.

Model *studije predavanja* započinje etapom u kojoj se učitelji okupljaju i definiraju konačne ciljeve koji će osigurati učenje i razvoj učenika (Lewis, Perry i Murata, 2006). Stigler i Hybert (1999) identificirali su sljedeće etape *studije predavanja*: definiranje problema, planiranje predavanja, provođenje istraživačkoga predavanja, refleksija i vrednovanje predavanja, revizija predavanja, ponovna provedba revidiranoga predavanja te razmjena mišljenja, ocjenjivanja i rezultata ponovno provedenoga predavanja. Najistaknutiji dio ovih etapa je istraživačko predavanje gdje se planirano predavanje izvodi i kontrolira u stvarnom razrednom okruženju. To su predavanja koja se održavaju uz učenike nastavnika koji sudjeluju u istraživanju, razlikuju se od svakodnevnoga razrednog okruženja, imaju jedinstvena svojstva i održavaju se u ozračju stvarne učionice (Lewis i Tsuchida, 1998). Nakon što su ciljevi predavanja definirani u pripremi, nastavnici pripremaju plan nastavnoga sata. Umjesto stvaranja dobrog plana, cilj ove faze je procijeniti kako učenici uče i kako je usvojen pristup učenju, odnosno istražiti određeno pitanje o učenju (Murata, 2011).

Model studije predavanja stavlja istraživačko predavanje u središte procesa (Murata, 2011). Istraživanja pokazuju da japanski učitelji tvrde da su primijenjeni sati najvažniji doprinos njihovom stručnom usavršavanju (Murata, 2011; Murata i Takahashi, 2002). U istraživačkim predavanjima nastavnici vide modele obrazovanja i kako oni utječu na učenje učenika (Murata, 2011). Istraživačka predavanja služe razvijanju prakse u učionici, diseminaciji novih sadržaja i odgovarajućih pristupa, povezivanju učioničke prakse sa sveobuhvatnijim postignućima, proučavanju sukobljenih ideja i pružanju drugačijega pristupa, oblikovanju nacionalne politike te procjeni, odnosno uvažavanju uloge obrazovanja u učionici (Lewis i Tsuchida, 1998). Istraživačka se predavanja održavaju u stvarnim razredima i pružaju nastavnicima posebnu priliku za učenje, što im je inače onemogućeno unutar razvijene zajednice stručnoga usavršavanja (Murata, 2011). Provedba nastave u učionici promatra se kao cjelina, a ne kao gledanje pojedinačnih dijelova videozapisa ili čitanje pojedinačnih dijelova knjiga o obrazovnom procesu (Murata, 2011).

Slika 1.

U modelu *studije predavanja*, u kojem se primjenjuje ciklički pristup jer se ciklus ponavlja nekoliko puta, nastavnici imaju priliku raspravljati o procesu učenja i kako poučavanje utječe na proces učenja (Murata, 2011). Ova kružna struktura prikazana je u nastavku (slika 1). Činjenica da priroda modela pokazuje geometrijsko razmišljanje nastavnika i daje im priliku stvoriti okruženje u kojem mogu koristiti međusobno geometrijsko razmišljanje bila je razlog zašto je model *studije predavanja* bio poželjan u ovom istraživanju.

Istraživačka skupina

Istraživačka skupina ove studije sastojala se od dobrovoljaca, nastavnika matematike iz različitih srednjih škola u Aydınu, Turska. Ova je studija provedena s ukupno osam nastavnika; troje nastavnika bilo je uključeno u ogledni proces studije, a petero u proces implementacije. U oba procesa, studija temeljena na dobrovoljnom sudjelovanju provedena je u radu sa srednjoškolskim nastavnicima matematike čiji su tjedni rasporedi bili prikladni za vremenski okvir provođenja studije. Učitelji koji su sudjelovali u oglednoj studiji bili su isključeni iz glavne implementacije.

S ciljem odabira nastavne skupine za provođenje glavne studije, istraživač je održao sastanke s nastavnicima svih srednjih škola Središnjeg okruga koji su u ovoj fazi bili obaviješteni o procesu istraživanja. U međuvremenu su identificirani nastavnici koji su bili voljni dobrovoljno sudjelovati u studiji te je i s njima održan kontaktni sastanak. Tim je nastavnicima (njih četrnaestero) objašnjen model studije predavanja koji će se koristiti u istraživanju za stručno usavršavanje nastavnika i teorijske osnove *geometrijskih mentalnih navika* koje će se koristiti kao osnova za razvoj geometrijskoga razmišljanja. Osim toga, navedeni su i neki primjeri koji se odnose na oglednu studiju, a predstavljen je i nacrt radnoga rasporeda studije. Od tih četrnaestero nastavnika, petero je identificirano kao sudionici studije te su dobili dodatna objašnjenja o tome kako bi trebali sudjelovati u seminarima učitelja, koji će biti organizirani u skladu s krajnjim ciljevima studije, sudjelovati u aktivnostima studije predavanja koje će se provoditi i obavljati vlastite zadatke u skupini tijekom određenoga vremena. Istraživačka skupina informirana je o provedbi i pismeno obaviještena o zadatcima koje će obavljati i odgovornostima koje će preuzeti prije i tijekom studije. Zatim je petero srednjoškolskih nastavnika matematike (koji su prihvatili te odgovornosti) zajedno s istraživačem pripremilo detaljniji raspored rada u okviru svojih nastavnih planova.

Nastavnici koji su sudjelovali u istraživanju dobili su kodna imena, na temelju privatnosti primjerice, *Nastavnik A, T, O, M i S*. Nastavnik T i nastavnik C radili su u školi u kojoj je provedena implementacija, a nastavnici O, A i M radili su u drugim školama. Osim nastavnika O, svi nastavnici koji su sudjelovali bili su prvostupnici obrazovanja; nastavnik O bio je prvostupnik znanosti, s pedagoškim certifikatom i magisterijem u matematici (specijalizacija u algebri). Osim toga, nastavničko iskustvo sudionika *studije predavanja* bilo je sljedeće: za nastavnika A i nastavnika M - 2 godine, za nastavnika S - 5 godina, za nastavnika T - 7 godina te za nastavnika O - 15 godina.

Proces provedbe

Proces provedbe studije izvršen je u dvije faze: ogledna shema i glavna implementacija. Tijekom ogledne studije (seminari su trajali pet tjedana, a *studija predavanja* osam tjedana) testirane su aktivnosti korištene na seminarima, razmatrana su mišljenja stručnjaka iz ovoga područja, a istraživač koji će po prvi put primijeniti ovaj model dobio je relevantno iskustvo.

U prvoj etapi glavne studije, nastavnički seminari održani su zajedno s nastavnicima u periodu od pet tjedana (dva puta tjedno , a svaki je seminar trajao oko tri sata). Na ovim seminarima nastavnicima je objašnjena svrha ove studije, model studije predavanja i *okvir geometrijskih mentalnih navika*, raspravljalo se o konceptu geometrijskoga razmišljanja, riješilo neke geometrijske zadatke te su održane rasprave o pretpostavkama nastavnika o tome kako će implementacija teorijskoga okvira utjecati na stvarne učenike, tražeći od nastavnika da predviđaju odgovore učenika na te zadatke (slika 2).

Slika 2.

U sesijama *studije predavanja*, nastavnici su zamoljeni da poučavaju gradivo koje pokriva ciljeve vezane uz *trokute i mjerenja trokuta* u području geometrije za 8. razred srednje škole u jesenskom i proljetnom polugodištu. Za razliku od ogledne studije, sate su pripremali nastavnici pojedinačno u školama u kojima rade te su i kontrolirani na kraju ovoga procesa s ciljem procjene utjecaja razvoja geometrijskoga razmišljanja nastavnika na njihovo individualno učenje (odn. na sadržaje koje su poučavali u vlastitim školama, bez obzira na *studiju predavanja*). Ova studija uključivala je samo procese *studije predavanja* nastavnika.

Plan studije i organizacija studije predavanja

U procesu određivanja radnoga rasporeda *studije predavanja* razmotrili smo kružnu strukturu modela i raspored sati koje nastavnici moraju bez odgađanja odraditi u školama u kojima rade. Na početku polugodišta, istraživač je zatražio od nastavnika sudionika da predaju nacrte rasporeda nastave, a nakon intervjua s ravnateljima škola, raspored nastave bio je sastavljen prema danima i satima koji su bili pogodni za učitelje kako bi mogli doći i promatrati istraživačka predavanja. U skladu sa strukturom modela, u slučaju neučinkovitoga istraživačkog predavanja provedenog nakon donošenja odluka o procesu planiranja, uključena je druga skupina učenika 8. razreda iste škole unutar vremenskog okvira studije i prema dostupnosti učitelja, s ciljem primjene modela u novom razrednom odjelu.

Tijekom ovoga procesa, nastavnici su svakog petka i ponedjeljka putovali u školu u kojoj je provođena implementacija modela, a ostalim danima održavali su planiranja i rasprave u skladu s rasporedom tečaja. Nastavnici su ispunili uvjete *studije predavanja* tako što su se okupili četiri puta tijekom tjedana kada su osjetili potrebu za ponavljanjem ili nastavljanjem istraživačkoga predavanja. U skladu sa studijama koje su nastavnici proveli i odlukama koje su donijeli, strukture 1., 2., 5., 6. i 7. *studije predavanja* primijenjene su u tri faze: sastanak za planiranje, istraživačko predavanje i rasprava. S obzirom na činjenicu da nakon sastanka za planiranje, istraživačkoga predavanja i sastanka za raspravu, zakazano vrijeme nije bilo dovoljno za planirane teme, učitelji su osjetili potrebu za održavanjem istraživačkoga predavanja koje je nastavak studije predavanja te je to predavanje raspravljeno u 3. i 4. razredu.

Okruženje

U procesu istraživanja, nastavnički seminari i sastanci *studije predavanja* (planiranje i vrednovanje) održani su u konferencijskoj dvorani unutar Sveučilišta gdje su na raspolaganju bili: bijela ploča, računalo s različitim vrstama geometrijskih programa, projektor povezan s računalom, alati potrebni za sate geometrije (šestari, ravnala, kutomjeri, geometrijski štapići, jedinične kocke, tangrami itd.) i drugi materijali kao što su papir, škare, kartoni i ljepila koji se mogu koristiti u pripremi. Pobrinuli smo se da nastavnici mogu udobno raditi kao skupina u okruženju u kojem mogu sjediti licem u lice za pravokutnim stolom.

U učionici u kojoj su provedena istraživačka predavanja dodane su još dvije klupe kako bi se osiguralo da prirodno okruženje nije narušeno, a da nastavnici mogu promatrati izvedbu *studije predavanja* u učionici. U učionici u kojoj je održano predavanje bilo je dostupno računalo, projektor i alati potrebni za sat geometrije.

Prikupljanje i analiza podataka

U ovoj studiji podatci su prikupljeni pomoću videozapisa, terenskih bilješki istraživača, obrazaca nastavničkih promatranja, istraživačkoga dnevnika te nastavničkih dnevnika seminara i sesija studije predavanja (planiranje, istraživačko predavanje, procjena i sastavljanje izvješća) provedenih u kontekstu istraživanja. Ova je studija kao metodu prikupljanja podataka koristila triangulaciju podataka pomoću recenzija intervjua, promatranja i dokumenata.

Priroda modela studije predavanja je da nastavnici, zajedno s cijelim timom, prate istraživačko predavanje koje su planirali u stvarnom razrednom okruženju kako bi ga analizirali na sastanku za raspravu. Tijekom ove studije nastavnici su zamoljeni da na kraju nastave popunjavaju nastavničke promatračke obrasce koji su uključivali sastavnice i podsastavnice geometrijskih mentalnih navika koje je pripremio istraživač kako bi se osiguralo da nastavnici mogu u potpunosti razmatrati geometrijsko razmišljanje u svojim opažanjima i tijekom rasprave. Osim toga, pobrinuli smo se da se uzme u obzir i stručno usavršavanje, dajući svakom nastavniku dnevnik i tražeći od njih da napišu svoje ciljeve sastanaka za planiranja i rasprave, kao i ono što uviđaju kao nedostatke u vlastitom radu s obzirom na određenu temu. Zajedno s ovako dobivenim podatcima, za provjeru podataka dobivenih iz videozapisa korištene su i terenske bilješke istraživača i istraživački dnevnici.

U modelu *studije predavanja* nastavnici pripremaju izvješće nakon završetka predavanja, uključujući ono što je učinjeno i što trebaju odnosno trebaju odnosno ne trebaju činiti, a što mogu ostvariti u kasnijoj fazi, uvažavajući njihove savjete. Ovo se izvješće naziva i refleksivno izvješće ili konačno izvješće. Mišljenja nastavnika, kao i odluke o reviziji vezane za plan predavanja, postala su očitija istraživaču uz pomoć ovih izvješća.

Tijekom istraživanja pokušali smo osigurati verifikaciju podataka prikupljenih iz drugih izvora provođenjem polustrukturiranoga intervjua s nastavnikom koji je vodio predavanje nakon svakog istraživačkoga predavanja. Napravljene su i audio i videosnimke intervjua. Tijekom ovih razgovora nastavnici su postavili sljedeća pitanja: *Je li istraživačko predavanje postiglo svoj cilj?*, *Što je prošlo dobro, a što nije?*, *Koje stvari nisu išle prema planu?*, *Koje ste geometrijske navike pokušali promovirati?*, *Koje ste dokaze geometrijskih navika promatrali tijekom predavanja?* i *Postoje li neke točke koje treba revidirati ili ne*?

U ovome je istraživanju analiza podataka provedena kao „kontinuirani proces analize" za vrijeme same provedbe te kao „retrospektivna analiza i proces izgradnje modela" nakon provedbe (Steffe i Thompson, 2000; Cobb, 2000; Simon, 2000).

Provedene analize, zajedno s odgovorima pojedinaca, tvore temelj za nepripremljene i planirane intervencije, kao i za interakcije koje omogućuju stjecanje više znanja, testiranje hipoteza i promicanje budućega razvoja. Osim toga, znanje, djelovanje i karakteristike pojedinaca temeljili su se na razvoju i mijenjanju modela koji je izradio istraživač (Simon, 2000). Međutim, retrospektivne analize nastavljaju razvijati deskriptorski model matematičkoga razvoja opetovanim pregledom većine podataka i revidiranjem svih povezanih zapisa u pažljivo strukturiranom obliku (Simon, 2000). Nadalje, one usmjeravaju proces stvaranja zapisa u vezi s razvojem hipoteza istraživača i razvojem ponašanja, mišljenja i djelovanja pojedinaca tijekom razredne prakse (Molina, Castro i Castro, 2007).

U ovom je istraživanju analiza videozapisa sastanaka i istraživačkih predavanja koje su proveli nastavnici u procesu *studije predavanja* provedena u sedam etapa: pažljivo gledanje videopodataka – opisivanje videopodataka – otkrivanje kritičnih događaja – transkripcija – kodiranje – izgradnja priče – sastavljanje pripovjedi.

Nakon analize podataka proces je modeliran stvaranjem predloška za svaki sastanak. Kako bi se osigurala valjanost i pouzdanost analize podataka, podatke su analizirani dvojica stručnjaka u području poučavanja matematike pomoću metode triangulacije istraživača čime je osigurana vanjska valjanost istraživanja.

U primijenjenim *studijama predavanja*, a s ciljem utvrđivanja razvoja geometrijskoga razmišljanja nastavnika, podatci dobiveni iz različitih izvora (videosnimke, terenske bilješke, bilješke nastavnika, dnevnici nastavnika i refleksivna izvješća o studiji predavanja) tijekom sastanaka provedenih s nastavnicima, analizirani su pomoću teorijske strukture geometrijskih mentalnih navika (*GHoM*), koja se sastoji od četiri geometrijske navike već objašnjene u teorijskom okviru, odnosno *rasuđivanje o odnosima*, *generalizacija geometrijskih ideja*, *istraživanje invarijanti* te *balansiranje istraživanja i razmišljanja*.

Kako bi se osiguralo da je ovo istraživanje konačno, formirana je dugoročna interakcija s istraživačkom skupinom te je korištena triangulacija pomoću različitih alata za prikupljanje podataka. Zaključivanje vezano za prikupljene podatke u zapisnicima sa sastanaka za raspravu, testirano je provjerom (u nastavničkim promatračkim bilješkama) jesu li geometrijske navike koje su naveli na sastancima za raspravu definirane tijekom predavanja. Tijekom istraživanja pokušali smo osigurati da se ti podatci mogu prenositi predstavljanjem kroz opise i izravna citiranja. Podatke i kodiranje samostalno i nezavisno provjeravalo je dvoje nastavnika matematike kako bi se osigurala konzistentnost istraživanja i analiza potvrde. Nadalje, prihvaćena su mišljenja stručnjaka u ovom području o provedenim aktivnostima, a sve provedbe tijekom procesa zabilježeni su na videozapisima.

Rezultati

U kontekstu istraživanja, 1., 2., 5., 6. i 7. *studija predavanja* uključivala je tri etape: planiranje (P), istraživačko predavanje (IP) i rasprava (R). 3. i 4. studija predavanja sastoje se od pet etapa zbog potrebe provedbe novoga IPa i novoga sastanka za raspravu (R), koji su zapravo bili nastavak istraživačkoga predavanja nakon etapa planiranja i rasprave o istraživačkom predavanju (slika 3).

Slika 3.[2]

Nastavnici su, posebice u prve dvije studije predavanja, istaknuli da se nisu mogli naviknuti na prirodu metode i da su, kada su započeli etapu rasprave, shvatili kako nisu proveli nikakva istraživanja na sastancima za planiranje kako bi ispravili svoje pogreške, uklonili nedostatke i razvili svoje geometrijske navike. U 3. i 4. *studiji predavanja*, iako su nastavnici pripremili detaljne planove nastavnoga sata, nisu ih mogli ostvariti unutar vremenskog okvira koji su očekivali tijekom primjene IPa. U ovome se istraživanju rezultati koji se odnose na razvoj geometrijskih navika nastavnika temelje na dva pokazatelja: prazninama u nastavničkom znanju o geometrijskim navikama te na navikama koje su učitelji promicali tijekom nastave.

Tablica 2

Na primjer, u prvoj studiji predavanja, činjenica da je nastavnica S imala nedovoljno znanje o sastavnici *provjera dokaza o učincima* shvaćena je na osnovi njezinih izjava navedenima u nastavku:

S: *I, hmm, onaj prvi trokut koji su učenici nacrtali u bilježnicama, mislila sam da je to provjera dokaza o učincima.…*
R: *To ste mislili jer su sami nacrtali trokute?*
S: *Da.*
R: *To ste pomislili nakon što su nacrtali koji trokut?*
S: *Pa, znate, pokušali su sami crtati... razmišljala sam o onima koje su sami nacrtali i kakav bi bio učinak njihovog crtanja...*
R: *Dakle, mislite li da bi bilo prikladno da je 3 ili 4 [*odnosi se na duljinu stranice*]? Na primjer, koja bi trebala biti duljina 3. stranice?*
S: *Tako je. I još jedna stvar:* nacrtala je trokute s duljinama stranica 7-10-11, 8-9-3...

Sljedeći primjer može se dati kao primjer nedostatka znanja vezan uz činjenicu da iako se promicala sastavnica *uporabe vještina prostornog rasuđivanja*, vještine *simetrije i proporcionalnoga rasuđivanja* nisu se rabile u istoj studiji predavanja:

R: *Sada ću prijeći na drugi primjer. Na koje načine ovo pitanje (zadatak 4) može otkriti geometrijsko razmišljanje?...*

Zadatak 4

A: *To je samo jedan oblik... Usredotočujući se na odnos između dijelova?*
M: *Tako je, odnos između dijelova.*
R: *Točno, što još?*
A: *Ona rabi vještine prostornog rasuđivanja.*
R: *Što mislite pod „vještinama prostornog rasuđivanja"?*
A: *Ona razdvaja dva, a zatim razdvaja još dva od ta dva* [odnosi se na trokute ABD i BCD], *a zatim ih spaja.*

[2] P: Sastanak za planiranje, RL: Istraživačko predavanje, D': Sastanak za raspravu, RL': Nastavak istraživačkoga predavanja, D': Nastavak sastanka za raspravu

Kao što se može shvatiti iz izjave nastavnice, budući da je spomenula prisutnost sastavnice *uporaba vještina prostornog rasuđivanja* temeljenu na nalazima koji se nisu odnosili na ovu sastavnicu, ova je situacija pokazala da nije imala spoznaje o toj temi.

Ona je u ovo istraživanje uključena u kontekstu navika, odnosno načina na koji su nastavnici završili proces – promicanjem određenih geometrijskih navika korištenih tijekom svake *studije predavanja.*

Mentalna navika rasuđivanja o odnosima

Tablica 3 pokazuje koje su sastavnice *rasuđivanja o odnosima* nastavnici pokušali promicati kroz studije koje su proveli u svakoj etapi procesa *studije predavanja*. Kako bi se utvrdio nedostatak znanja nastavnika o bilo kojoj sastavnici u odgovarajućim *studijama predavanja*, kôd nastavnika označen je crvenom bojom u donjoj tablici. Osim toga, njihovi kodovi označeni su podebljanim slovima kako bi naglasili jesu li promicali bilo koju sastavnicu u odgovarajućim *studijama predavanja*.

U analizi ove tablice zamijećeno je da bez obzira na to kako su nastavnici pokušali promicati sastavnice *usredotočenja na odnose između pojedinih likova* i *usredotočenja na odnose između dijelova jednog lika*, kao u 1. *studiji predavanja*, svejedno su imali poteškoća u promicanju sastavnice *uporaba posebnih vještina rasuđivanja* u usporedbi s drugim sastavnicama.

U postupku analize videozapisa sastanka za planiranje 1. *studije predavanja* utvrđeno je da nastavnici nisu izravno uključili ni jednu geometrijsku naviku. Istraživači su analizirali aktivnosti koje su planirali nastavnici u kontekstu geometrijskih navika te ih dodali u tablicu. Prilikom ocjenjivanja tablice 3, uzevši u obzir nedovoljno znanje nastavnika, može se primijetiti da nakon pohađanja seminara nastavnici nisu imali prevelik nedostatak znanja o mentalnoj navici *rasuđivanja o odnosima* i da su nastojali intenzivno promicati tu sastavnicu. Zamijećeno je da su nastavnici imali nedostatak znanja/ pogrešnu predodžbu o sastavnici *uporabe vještina prostornog rasuđivanja* samo u prve dvije *studije predavanja* i da su uklonili svoje nedostatke u kasnijim *studijama predavanja*. Osim toga, može se primijetiti da su, kako se proces razvijao, pokušali unaprijediti sve ove tri sastavnice u studijama koje su proveli.

Tablica 3

Iz onoga što su nastavnici izjavili na sastanku za raspravu o studiji predavanja, može se shvatiti da ih je aktivnost koju su pripremili potaknula da istaknu rasuđivanje i time potiču mentalnu naviku *rasuđivanja o odnosima*, omogućujući učenicima da koriste vještinu *proporcionalnoga rasuđivanja*:

***A**: Zapravo, ovdje se može vidjeti i sastavnica „uporabe vještina prostornog rasuđivanja"?*

***R:** Želite reći da je možemo promatrati?*

***A**: Nisam siguran.*

***M**: Već sam vam rekao da se može promatrati.*

***A**: Upravo sam shvatio, da, izgleda da se može promatrati.*

R: *Na primjer, na koji način rasuđivanja mislite kada govorite o „prostornom rasuđivanju"?*
A*: Imam taj osjećaj, mislim, kao da se tamo nalazilo; mislim, rekao je da je bilo tamo.*
O*: Vještine prostornog rasuđivanja?*
A*: Mislim, rekao je da je bilo tamo.*
O*: „Prostorno rasuđivanje" je... kad koristimo proporcionalno rasuđivanje između dva ili više od dva geometrijska oblika... On će proporcionalno rasuđivati između dva... Stranica jednog od trokuta je 1,5 puta dulja od stranice drugih trokuta.*

Kao što je vidljivo iz ovih komentara nastavnika, i procesi analize u geometrijskom razmišljanju i intelektualiziranje geometrijskih navika u sastanku za raspravu razvili su mentalnu naviku *rasuđivanja o odnosima* kroz uporabu posebnih vještina rasuđivanja.

Mentalna navika generalizacije geometrijskih ideja

Sastavnice koje su nastavnici promicali o mentalnoj navici *generaliziranja geometrijskih ideja* u studijama koje su proveli u procesu studija predavanja prikazane su u tablici 4. Ako uzmemo u obzir nedostatke znanja učitelja o mentalnoj navici *generaliziranja geometrijskih ideja*, može se primijetiti da su nastavnici imali previše praznina u znanju, osobito s obzirom na sastavnicu *uporabe poznatih slučajeva* u prve tri *studije predavanja*. Kada su razmatrane druge sastavnice mentalne navike *generaliziranja geometrijskih ideja*, primijećeno je da su nastavnici više promicali te sastavnice u 5., 6. i 7. *studiji predavanja*.

Nakon rasprave o planu aktivnosti, nastavnik O i nastavnik A održali su još jednu raspravu o tome promoviraju li oni situacije koje će potaknuti učenike da generaliziraju pomoću sekundarnih elemenata trokuta:

A*: Na primjer, dat ćete jedan običan trokut, zar ne? Drugim riječima, četiri... mislim, koliko ja razumijem, njegova visina je viša od ove... koliko je rekao naš prijatelj... to jest, njegova težišnica je iznad ove... Dakle, ovo me zbunjuje: on bi trebao moći nacrtati težišnicu u 3-4 trokuta. Ako je može nacrtati u raznostraničnom trokutu, trebao bi je moći nacrtati i u jednakokračnom i jednakostraničnom trokutu...*

Pri promatranju mentalne navike *generaliziranja geometrijskih ideja* u kontekstu *uporabe poznatih slučajeva*, primijećeno je da su nastavnici pokušali promovirati ovu sastavnicu već od 1. *studije predavanja*. Međutim, uz vodstvo istraživača, zbog prevelikog nastavničkog nedostatka znanja o ovoj sastavnici u prve tri studije predavanja, uočeno je da se ti nedostatci počinju uklanjati od 5. *studije predavanja*. Zamijećeno je da nastavnici nisu imali nedostatak znanja o *uporabi pretpostavljenih pojednostavljujućih uvjeta*, što je još jedna sastavnica mentalne navike *generalizacije geometrijskih ideja*, niti su se trudili previše promicati ovu sastavnicu prije 5. *studije predavanja*. Nadalje, prepoznato je da neki nastavnici nisu imali dovoljno znanja o ovoj sastavnici kada su krenuli sa 7. *studijom predavanja*.

U kontekstu promicanja sastavnice *pronalaženja kompletnih nizova rješenja ili općih pravila*, pri proučavanju *studija predavanja* koje su nastavnici proveli, zabilježeno je da su svi nastavnici u 2. i 6. *studiji predavanja* i gotovo svi nastavnici u 3. i 5. *studiji predavanja* pokušali promicati ovu sastavnicu. Nadalje, zabilježeno je da osim nastavnika T (u 1. *studiji predavanja*) i nastavnika S (u 4. *studiji predavanja*) ni jedan nastavnik nije imao nedostatak znanja o toj sastavnici.

Tablica 4

Mentalna navika istraživanja invarijanti

Sastavnice koje su nastavnici promicali vezane za mentalnu naviku *istraživanja invarijanti* u svakoj fazi procesa *studije predavanja* navedene su u tablici 5. Kako bi se utvrdio nedostatak znanja nastavnika o bilo kojoj sastavnici u odgovarajućim *studijama predavanja*, kôd nastavnika označen je crvenom bojom u donjoj tablici. Kada je tablica 5 pregledana, moglo se primijetiti da su nastavnici pokušali promicati sve sastavnice već od 1. *studije predavanja*. Kada su proučavane studije nastavnika o razvoju mentalne navike *istraživanja invarijanti*, uočeno je da su svi nastavnici imali previše nedostatka znanja o ovoj sastavnici u prve tri *studije predavanja*. Također je zabilježeno da se njihov nedostatak znanja smanjio te su promicali obje sastavnice od 4. *studije predavanja*.

Kada je nedostatak znanja nastavnika o sastavnicama ove navike procijenjen zajedno s istraživanjima koja su proveli kako bi ih promicali, moglo se reći da su pokušavali ukloniti nedostatak znanja o mentalnoj navici *istraživanja invarijanti* iz 4. *studije predavanja* i promicati tu naviku do kraja istraživanja.

Nadalje, zabilježeno je da nastavnica S nije imala dovoljno znanja o koraku *provjera dokaza o učincima* sastavnice *istraživanja invarijanti*.

Iz njezinih komentara u nastavku može se shvatiti da je pomiješala ovaj korak s korakom istraživanja u sastavnici *balansiranje istraživanja i razmišljanja*, što je još jedna geometrijska navika:

S: *I, hm, u prvom trokutu koji su studenti nacrtali u bilježnicama, mislila sam da je to, ovaj, provjera dokaza o učincima pri istraživanju invarijanti ... također...*

R: *To ste mislili jer su sami nacrtali trokute?*

S: *Da.*

R: *To ste pomislili nakon što su nacrtali koji trokut?*

S: *Pa, znate, pokušali su sami crtati... razmišljala sam o onima koje su sami nacrtali i kakav bi bio učinak njihovog crtanja...*

R: *Dakle, mislite da bi bilo prikladno da je 3 ili 4? Je li to bilo kao, na primjer, koliko treba iznositi duljina 3. stranice?*

S: *Tako je.*

Tablica 5

Mentalna navika balansiranja istraživanja i razmišljanja

Sastavnice koje su nastavnici promicali o mentalnoj navici *balansiranja istraživanja i razmišljanja* u studijama koje su proveli u procesu studija predavanja prikazane su u tablici 6. Kako bi se utvrdio nedostatak znanja nastavnika o bilo kojoj sastavnici u odgovarajućim *studijama predavanja*, kôd nastavnika označen je crvenom bojom u donjoj tablici.

Primijećeno je da su nastavnici nastojali da se u prve tri *studije predavanja* istakne sastavnica *postavljanja istraživanja u prvi plan*, a u posljednje tri *studije predavanja* sastavnica *postavljanja konačnih ciljeva u prvi plan*. Uočeno je da, iako su nastavnici pridali veliku važnost sastavnici *postavljanja konačnih ciljeva u prvi plan*, nisu dovoljno razumjeli sastavnicu *postavljanja istraživanja u prvi plan*.

Slika 5.

Tablica 6

Uočeno je da su nastavnici koristili gore navedeni zadatak (slika 5) (*Što može biti duljina stranice A u ovom trokutu?*) na početku procesa *studije predavanja* koju su proveli. Iz sljedećih komentara nastavnika može se shvatiti da su učenici koristili ovaj problem kako bi testirali nejednakost trokuta na primjerima *konstrukcija-dekonstrukcija trokuta* te da su nastavnici miješali korak komponente *balansiranje istraživanja i razmišljanja* koji *postavlja istraživanje u prvi plan* s korakom *provjere dokaza o učincima*:

***R**: Ovdje smo* [slika 5] *pitali koje se vrijednosti mogu dodijeliti A. Postoji li nešto u vašim bilješkama što biste željeli dodati? Možda Vi, nastavniče M? Koje smo sastavnice o geometrijskom razmišljanju ovdje koristili?*

***M**: Vjerojatno „provjera dokaza o učinku”, valjda...*

***R**: Je li to moguće? Ako je tako, na koje načine to može biti tako?*

***M**: Ponovno pokušavaju iskoristiti tu vezu u primjerima.*

Kada su analizirani dokazi koji ukazuju na nastavničku mentalnu naviku *balansiranja istraživanja i razmišljanja*, uočeno je da, iako su do 5. etape pokazivali nedostatak znanja, pokušali su promicati tu naviku već od 1. etape.

Razvoj geometrijskoga razmišljanja nastavnika odražava se u smanjenju razine nedostatka znanja, kao i u njihovim komentarima o ovoj sastavnici, osobito od 5. *studije predavanja* nadalje. Na primjer, nastavnik T objasnio je razliku između sastavnica *postavljanja konačnih ciljeva u prvi plan* i *postavljanja istraživanja u prvi plan* putem metafore:

„Sada zamislite da je brod potonuo sve do morskog dna. Provedba istraživačke studije broda je „istraživanje” , no ponovno ronjenje za nečim što ste već istražili je „vaš cilj”. Kada kažemo da postoji brod na dnu mora, oni odmah počinju tražiti blago na dnu mora.”

Rasprava i zaključci

Geometrijske mentalne navike (Driscoll i sur., 2007) definiraju i objašnjavaju načine razmišljanja (*rasuđivanje o odnosima, generaliziranje geometrijskih ideja, istraživanje invarijanti te balansiranje istraživanja i refleksije*) koje i učenici i nastavnici trebaju koristiti kako bi bili uspješni u rješavanju geometrijskih zadataka. Zapravo, nije jednostavan proces utvrđivanja je li učitelj dobar u rješavanju geometrijskih problema nije jednostavan.

U procesu planiranja *studije predavanja* nastavnici trebaju predvidjeti moguće odgovore učenika i uzeti u obzir te detalje prilikom planiranja sata (Murata, 2011). U ovoj studiji, u prve dvije *studije predavanja*, nastavnici su istaknuli da se nisu mogli naviknuti na prirodu metode te da su nakon etape rasprave uvidjeli da nisu proveli nikakva istraživanja na sastancima planiranja kako bi ispravili svoje pogreške, uklonili svoje nedostatke i razvijali svoje geometrijske navike. U 3. i 4. *studiji predavanja*, nastavnici su pripremili detaljne planove nastavnoga sata, no nisu ih mogli ostvariti unutar vremenskoga okvira koji su očekivali tijekom primjene IPa (istraživačkoga predavanja). Ova situacija mogla bi proizaći iz činjenice da IP nije detaljno razrađeno u procesu planiranja postavljajući mišljenja učenika u prvi plan, kao što je naglasio Murata (2011).

Uočeno je da su nastavnici imali nedostatak znanja o mnogim navikama tijekom prve četiri *studije predavanja*, intervencije istraživača poduzete su kako bi se riješili nedostaci, a tijekom tog procesa došlo je do smanjenja razine nedostatka znanja učitelja o njihovim navikama, osobito tijekom *studije predavanja* nakon 4. ciklusa. Kada je nedostatak znanja nastavnika o toj temi istražen, primijećeno je da su 2. *studija predavanja* za *rasuđivanje o odnosima*, 4. *studija predavanja* za *generaliziranje geometrijskih ideja*, 3. *studija predavanja* za *istraživanje invarijanti* i *balansiranje istraživanja i razmišljanja* bile prekretnice. U ovom slučaju, s obzirom da je prekretnica za *studije predavanja* provedene za potrebe istraživanja bila, zapravo, 5. *studija predavanja*, pretpostavlja se da će planirano usavršavanje morati trajati najmanje pet tjedana. Osim toga, nastavnici koji će pohađati ovo usavršavanje vezano za održavanje *studija predavanja* u vlastitim školama usmjeravat će nastavnike koji pokušavaju postići model *studije predavanja* u svojim školama.

Zaključeno je da su nastavnici pokušali promicati sve sastavnice koje se odnose na mentalnu naviku *rasuđivanja o odnosima*, kako u planiranju i provođenju sata, tako i na raspravi o predavanju koje su održali. Međutim, činjenica da su nastavnici imali poteškoća u promicanju sastavnice *korištenja vještina prostornog rasuđivanja*, pokušavajući promicati sastavnice *usredotočenja na odnose između pojedinih likova* i *usredotočenja na odnose između dijelova jednog lika* već od 1. *studije predavanja* može se objasniti prirodom ove sastavnice. Budući da je ova sastavnica usko povezana s korištenjem proporcijskoga razmišljanja u geometriji i simetriji, može se povezati s činjenicom da ovaj sadržaj nije prikladan za korištenje tih pojmova.

U analizi tablice zamijećeno je da, iako su nastavnici pokušali promicati sastavnice *usredotočenja na odnose između pojedinih likova* i *usredotočenja na odnose između dijelova jednog lika* iz 1. *studije predavanja*, imali su poteškoća u promicanju sastavnice *uporaba posebnih vještina rasuđivanja* u usporedbi s drugim sastavnicama.

Moglo bi se reći da, iako su učitelji imali prevelik nedostatak znanja o sastavnici *uporaba poznatih slučajeva* mentalne navike *generaliziranja geometrijskih ideja*, pokušali su promicati mentalnu naviku *generaliziranja geometrijskih ideja* kroz njezine ostale sastavnice. Ova se situacija može smatrati znakom da su dokazi prikupljeni za mentalnu naviku *generaliziranja geometrijskih ideja* pokazali kako je ta mentalna navika hijerarhijske prirode, kao što navode Driscoll i sur. (2007).

Kod mentalne navike *istraživanja invarijanti* primijećeno je da, iako nastavnici nisu mogli promicati oba načina rasuđivanja tijekom prve tri *studije predavanja*, počeli su ih promicati od 4. *studije predavanja* nadalje. Ova situacija mogla bi proizlaziti iz činjenice da su učenici u prve tri lekcije „izgradili glavne i sekundarne elemente trokuta." Stoga, Erduran i Yeşildere (2010) u svojoj studiji o „nastavničkoj izgradnji geometrijske strukture pomoću šestara i ravnala" utvrđuju da se lekcije poučavaju tako da je nastavnik u središtu pozornosti, a učenici pokušavaju slijediti upute učitelja kroz mehaničko učenje. Moguće je da je ova situacija prisilila nastavnike da razviju lekcije bogatije sadržajem u smislu geometrijskoga razmišljanja, kako su se kretali prema sljedećim temama i promicali geometrijske navike *dinamičkog razmišljanja i pretraživanja* i *provjere dokaza o učincima*.

Što se tiče mentalne navike *balansiranja istraživanja i razmišljanja*, koja je posljednja *geometrijska mentalna navika*, primijećeno je da nastavnici nisu imali prevelik nedostatak znanja. Moglo bi se reći da nastavnici nisu održali mnogo rasprava o ovome jer je tu naviku teško promicati, pa se stoga njihov nedostatak znanja možda ni neće potpuno razotkriti. Nastavnici trebaju osmisliti zadatke koji će potaknuti učenike da razviju svoje geometrijske vještine u procesu rješavanja problema, postavljajući istraživanje ili konačne ciljeve u prvi plan. Činjenica da se ova situacija počela otkrivati od 5. *studije predavanja* i dalje se može objasniti razvojem koji se također dogodio u nastavničkom poznavanju područja. Ako uzmemo u obzir da znanje nastavnika iz područja geometrije ima značajan utjecaj na učenje geometrije kod učenika, gore navedena situacija ima ključnu ulogu (Lenhart, 2010; Clements, 2003).

Učinak modela *studije predavanja* vidljiv je kada se uzme u obzir činjenica da ako želimo razviti geometrijsko razmišljanje nastavnika, također je nužno razviti njihove geometrijske mentalne navike. S ciljem razvijanja načina na koji nastavnici rasuđuju o geometrijskom razmišljanju, treba ih suočiti sa složenim geometrijskim zadatcima, potaknuti ih da predviđaju moguće odgovore učenika koji se odnose na te zadatke i osigurati da raspravljaju o tim odgovorima sa svojim kolegama tijekom svojih istraživanja. Stoga je predloženo da model *studije predavanja* bude češći i rašireniji, prvenstveno s ciljem razvijanja nastavničkoga predmetnog znanja geometrije i geometrijskoga razmišljanja. Kako bi se to osiguralo, seminari na kojima se poučava ovaj model moraju biti uključeni u programe stručnoga usavršavanja. Tako se može jamčiti da će nastavnici u prvom slučaju, kao i generacije koje slijede, bolje rješavati geometrijske zadatke.

Potvrda

Ovo je istraživanje dio doktorske disertacije Denisa Özena pod vodstvom izv. prof. dr. Nilüfera Kösea.

Ovo je istraživanje podržala Komisija za istraživačke projekte Sveučilišta Anadolu pod brojem potpore 1308E325.