THE QUATERNION REPRESENTATION OF STATIC GRAVITATIONAL FIELD:
POISSON’S EQUATION
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ABSTRACT
The quaternions are numbers which have division algebra. This property advantages for physicists. So, Quaternions can be used in the each field of physics and, physical quantities can be represented by the quaternions. In this paper, a quaternionic equation which replaces to the two vector equations of static gravitational field is written in 4-dimensions. In addition, Poisson’s equation is also defined in quaternionic representation.

Key Word: Quaternion, Poisson’s Equation, Static Gravitational Field.

1. INTRODUCTION
Complex numbers were a hot subject for research in the early eighteen hundreds. An obvious question was that if a rule for multiplying two numbers together was known, what about multiplying three numbers? For over a decade, this simple question had bothered Hamilton, the big mathematician of his day.

Hamilton had found a long sought-after solution, it was 4-dimensional. One of the first things Hamilton did was get rid of the fourth dimension, setting it equal to zero, and calling the result a “proper quaternion”. He spent the rest of his life trying to find a use for quaternions. By the end of the nineteenth century, quaternions were viewed as an oversold novelty.

In the early years of this century, Prof. Gibbs of Yale found a use for proper quaternions by reducing the extra fluid surrounding Hamilton's work and adding key ingredients from Rodrigues concerning the application to the rotation of spheres. He ended up with the vector dot product and cross product we know today.

Today, quaternions are of interest to historians of mathematics. Vector analysis performs the daily mathematical routine that could also be done with quaternions.

Quaternions which are very useful numbers in the justification of the postulates in special relativity, quantum and classical mechanics as well as in solving high energy physics' problems can be used to representing of physical quantities. Some of them, for example, are Dimensional - Directional Analysis by a Quaternionic Representation of Physical Quantities (Arenada,1996), General Quaternion Transformation Representation for Robotic Application (Tan and Balchen,1993), and Quaternion Scalar Field is another example (De Leo and Rotelli, 1992).
2. QUATERNION ALGEBRA

Considering that the physical quantities of Newtonian mechanics are scalars or vectorials. The both types of quantities in the quadri-dimensional vectorial space of the quaternions is possible represented. Then a physical scalar(vectorial) quantity is represented by a scalar(vector) quaternion.

A quaternion is a quantity represented symbolically by $Q$ and defined by the equation (Özdaş and Özdaş, 1986):

$$Q = q_0 + q_1 \hat{\lambda}_0 + q_2 \hat{\lambda}_1 + q_3 \hat{\lambda}_2 + q_4 \hat{\lambda}_3 = [q_0, q_1, q_2, q_3]$$

where the real numbers $q_k$ denote the component of $Q$ relative to the unitary quaternion $\lambda_k$ ($k=0,1,2,3$). The scalar and vectorial parts of $Q$ are designed, respectively, by $(Q)_s$ and $(Q)_v$, and they are defined by

$$(Q)_s = q_0$$
$$(Q)_v = q_1 \hat{\lambda}_0 + q_2 \hat{\lambda}_1 + q_3 \hat{\lambda}_2 + q_4 \hat{\lambda}_3$$

A quaternion is a scalar(vector) quaternion if its vectorial(scalar) parts are equal to zero.

The unitary quaternions $\lambda_k$ ($k=0,1,2,3$) satisfy the Hamilton and Taif multiplication table (Arenada, 1996):

$$
\begin{array}{c|cccc}
\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
\hline
\lambda_0 & 1 & \lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_1 & \lambda_1 & -1 & \lambda_3 & -\lambda_2 \\
\lambda_2 & -\lambda_3 & \lambda_2 & -1 & \lambda_1 \\
\lambda_3 & -\lambda_1 & -\lambda_2 & \lambda_1 & -1 \\
\end{array}
$$

The quaternion conjugate $Q^*$ of a given quaternion $Q$ are defined as (Horn, 1987):

$$Q^* = q_0 \hat{\lambda}_0 - q_1 \hat{\lambda}_1 - q_2 \hat{\lambda}_2 - q_3 \hat{\lambda}_3 = [q_0, -q_1, -q_2, -q_3]$$

The product of two quaternions $Q$ and $P$ with components $q_k$ and $p_k$ ($k=0,1,2,3$) is given by (Chou, 1992):

$$QP = (q_0p_0 - q_1p_1 + q_2p_2 + q_3p_3)\hat{\lambda}_0 + (q_0p_1 + q_1p_0 - q_2p_3 + q_3p_2)\hat{\lambda}_1 + (q_0p_2 + q_2p_0 - q_3p_1 + q_1p_3)\hat{\lambda}_2 + (q_0p_3 + q_3p_0 - q_1p_2 + q_2p_1)\hat{\lambda}_3$$

(4)

It must be observed that this product is not commutative ($QP \neq PQ$). But the product of quaternion is associative (Harauz, 1990):

$$P(QR) = (PQ)R.$$  

(5)

The inverse $Q^{-1}$ of a quaternion $Q$ whose norm $N_Q$ is different from zero is given by

$$Q^{-1} = \frac{Q^*}{N_Q}.$$  

(6)

Where $N_Q = QQ^*$. The quotient between a quaternion $P$ and a quaternion $Q$ with $N_Q \neq 0$ is defined as (Tanışlı, 1995):

$$Q^{-1}P = \frac{PQ^*}{N_Q}.$$  

(7)

To each vector quaternion $P$ with components $(p_1, p_2, p_3)$ a vector $P$ of the Euclidean tridimensional space with components $(P_1, P_2, P_3)$ is associated reciprocally.

If $P$ and $Q$ are the vectors associated, respectively, of the quaternion vectors $P$ and $Q$, then the scalars and vectorial products of these vectors can be expressed as

$$PQ = (PQ)_s$$
$$PxQ = (PQ)_v.$$  

(8)

It must be observed that (Funda and Paul, 1988):

$$PQ = -PQ + (PxQ)$$

Quaternion notation of $\nabla$ operator in the Hamilton's quaternion can be written as:

$$\nabla = \hat{\lambda}_1 \nabla_i$$

(9)

Divergence and curl operators are expressed (De reli, 1992),

$$\nabla \cdot F(x) = -\nabla \cdot F(x) + \nabla \times F(x) = -\text{div} F(x) + \text{curl} F(x)$$

(10)

where "·" and "×" are dot and cross product of two vector quaternions, respectively, and Laplace operator can be defined as follows:

$$N(\nabla) = \nabla_i \nabla_i.$$  

(11)

3. STATIC GRAVITATIONAL FIELD

It is known from experiments that the gravitational field of a point particle of mass $M$ is given by

$$g = \frac{GM}{r^2} \hat{e}_r$$

(12)

where $\hat{e}_r$ is a unit vector drawn outward from the particle. The value of the gravitational constant is

$$G = 6.67 \times 10^{-8} \text{cm}^3\text{gm}^{-1}\text{sec}^{-2} = 3,42 \times 10^{-8} \text{ft}^3\text{slug}^{-1}\text{sec}^{-2}.$$

(13)

It is also known from experiment that the gravitational field has the algebraic properties of a vector. For example, let $P$ be a distance $r_1$ from mass $M_1$ and a distance $r_2$ from $M_2$ (Figure 1). If $\hat{g}_1$ and $\hat{g}_2$ are the gra-
Gauss divergence theorem, Eq.(19) can be converted to
\[ \int_{\Sigma} \nabla \cdot \mathbf{g} \, d\Sigma = -4\pi G \int_{\Sigma} \rho \, d\Sigma \tag{20} \]

This result is an identity applying to any arbitrarily chosen region of integration implying that
\[ \nabla \cdot \mathbf{g} = -4\pi G \rho \tag{21} \]

The mass density \( \rho \) is therefore a source function for the gravitational field. More properly it should be called a "sink function" since the lines of \( \mathbf{g} \) always converge toward the matter. Since \( \mathbf{g} \) is conservative, a second fundamental differential equation obeyed by \( \mathbf{g} \) is:
\[ \nabla \times \mathbf{g} = 0 \tag{22} \]

Dot and Cross products of two vectors are defined with a quaternion equation which is the quaternion product of two quaternions. If we use the rule of product of two vector(pure) quaternions which are \( \mathbf{V} \) and \( \mathbf{g} \) (Eq. 8), a quaternion equation can be written as follows:
\[ \nabla \cdot \left[ f(x) \mathbf{0}, 0, 0 \right] = 0 \tag{23} \]

Where \( f(x) \) is \( 4\pi G \rho \). This equation of quaternion replaces both of Eq.(21) and Eq.(22). In addition, the substitution of \( \mathbf{g} = -\nabla \Phi \) into Eq.(23) leads to
\[ \nabla \cdot \nabla \Phi + \left[ f(x) \mathbf{0}, 0, 0 \right] = 0 \tag{24} \]

showing that the basic quaternion equation satisfied by the scalar quaternion of gravitational potential is Poisson’s equation. If \( f(x) \) is zero in Eq.(24), The equation will be Laplace’s equation.

4. CONCLUSIONS

The static gravitational field is conservative and the divergence of which is Gauss’ Law for gravitation. We have written a simpler and general way to express the static gravitational field and Poisson’s equation as the product of two quaternions. This equation has the same form as the vector equations which was written for conservative field and Divergence theorem. Doing physics with quaternions has very easy, useful and compact representation. It is shown that physical quantities can be represented with quaternions. Also, the test for a conservative field can be done with operator quaternions.

In this study, The defining equations are quaternionic partial differential equations. One of them is Laplace’s equation, another is Poisson’s equation.
Four Maxwell equations in the electromagnetic theory can be written with two quaternion equations which are electrostatic field and magnetic field.

REFERENCES


